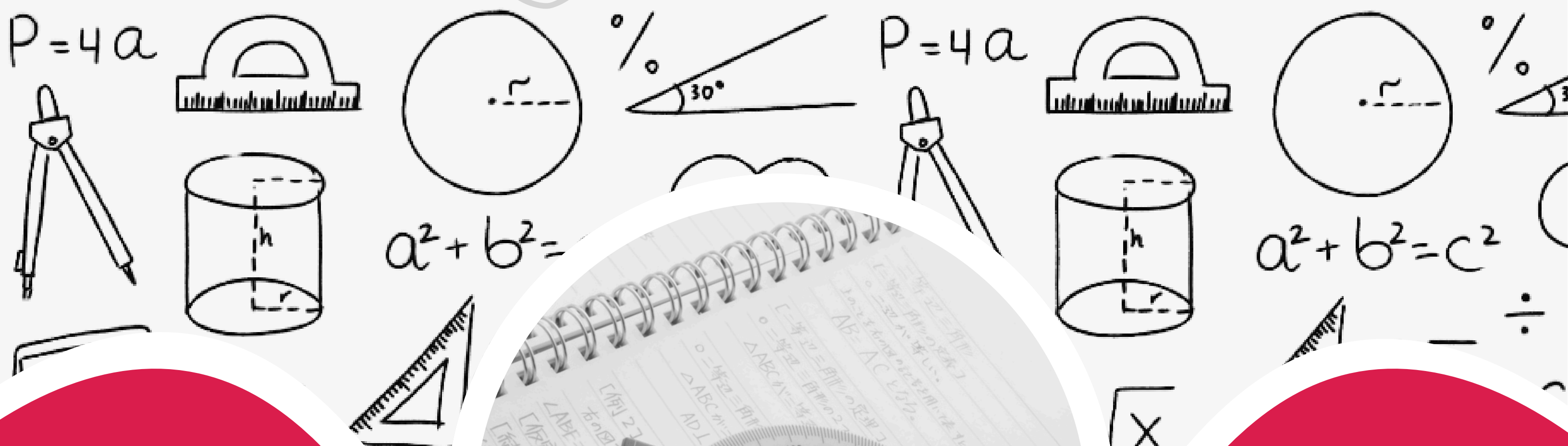
MOJZA

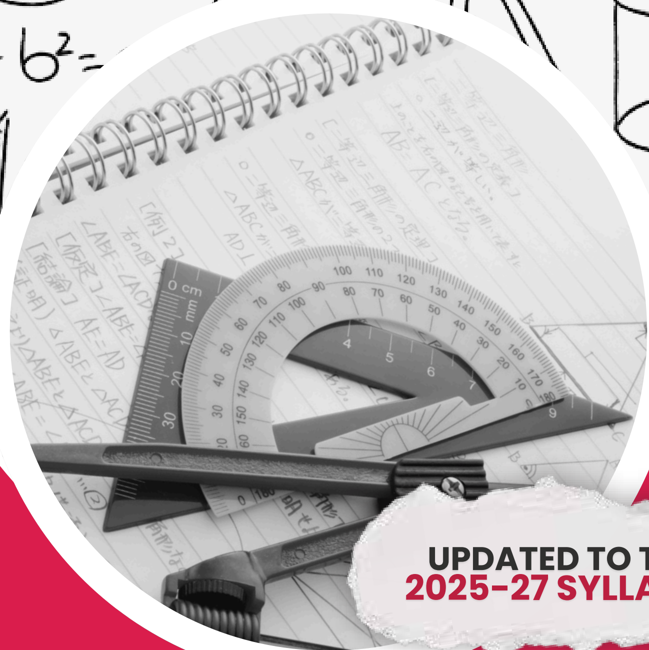
9231

AS Level

FURTHER MATHS

PAPER 1 - NOTES





UPDATED TO THE
2025–27 SYLLABUS

1st Edition

MOJZA

AS Level

Further Mathematics PAPER 1 NOTES

9231

BY TEAM MOJZA

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CONTENTS

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Roots of polynomial equations

Quadratic equation

➔for a quadratic equation $ax^2 + bx + c = 0$ with roots α , β

➔sum of roots = Σα = α + β = $-\frac{b}{a}$

➔product of roots = Σαβ = αβ = $\frac{c}{a}$

- Sum of roots

➔$S_n = \alpha^n + \beta^n$

➔$ax^2 + bx + c = 0$ becomes $aS_2 + bS_1 + cS_0 = 0$

➔$S_0 = 2$

➔$S_1 = \Sigma\alpha$

➔$S_2 = \Sigma\alpha^2 = (\Sigma\alpha)^2 - 2(\Sigma\alpha\beta)$

➔$S_{-1} = \Sigma\alpha^{-1} = \frac{\Sigma\alpha}{\Sigma\alpha\beta}$

➔for other values of n, $aS_2 + bS_1 + cS_0 = 0$ can be multiplied/divided by S_1, S_2 etc

➔when multiplied by S_1, it becomes $aS_3 + bS_2 + cS_1 = 0$

➔which can then be used to find the value of S_3

Cubic equation

➔for a cubic equation $ax^3 + bx^2 + cx + d = 0$ with roots α , β , γ

➔sum of roots = Σα = α + β + γ = $-\frac{b}{a}$

➔sum of pairs of roots = Σαβ = αβ + βγ + γα = $\frac{c}{a}$

➔product of roots = Σαβγ = αβγ = $-\frac{d}{a}$

- Sum of roots

➔$S_n = \alpha^n + \beta^n + \gamma^n$

➔$ax^3 + bx^2 + cx + d = 0$ becomes $aS_3 + bS_2 + cS_1 + dS_0 = 0$

➔$S_0 = 3$

➔$S_1 = \Sigma\alpha$

➔$S_2 = \Sigma\alpha^2 = (\Sigma\alpha)^2 - 2(\Sigma\alpha\beta)$

➔$S_{-1} = \Sigma\alpha^{-1} = \frac{\Sigma\alpha\beta}{\Sigma\alpha\beta\gamma}$
➔for S_3 , substitute known values in $aS_3 + bS_2 + cS_1 + dS_0 = 0$ and rearrange
➔for other values of n, $aS_3 + bS_2 + cS_1 + dS_0 = 0$ can be multiplied/divided by S_1, S_2 etc

Quartic equation

➔for a quartic equation $ax^4 + bx^3 + cx^2 + dx + e = 0$ with roots α , β , γ , δ
➔sum of roots = $\Sigma\alpha = \alpha + \beta + \gamma + \delta = -\frac{b}{a}$
➔sum of pairs of roots = $\Sigma\alpha\beta = \alpha\beta + \beta\gamma + \gamma\alpha + \alpha\delta + \beta\delta + \gamma\delta = \frac{c}{a}$
➔sum of triplets of roots = $\Sigma\alpha\beta\gamma = \alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta = -\frac{d}{a}$
➔product of roots = $\Sigma\alpha\beta\gamma\delta = \alpha\beta\gamma\delta = \frac{e}{a}$

- Sum of roots
➔$S_n = \alpha^n + \beta^n + \gamma^n + \delta^n$
➔$ax^4 + bx^3 + cx^2 + dx + e = 0$ becomes $aS_4 + bS_3 + cS_2 + dS_1 + eS_0 = 0$
➔$S_0 = 4$
➔$S_1 = \Sigma\alpha$
➔$S_2 = \Sigma\alpha^2 = (\Sigma\alpha)^2 - 2(\Sigma\alpha\beta)$
➔$S_{-1} = \Sigma\alpha^{-1} = \frac{\Sigma\alpha\beta\gamma}{\Sigma\alpha\beta\gamma\delta}$
➔for other values of n, $aS_4 + bS_3 + cS_2 + dS_1 + eS_0 = 0$ can be multiplied/divided by S_1, S_2 etc
➔when divided by S_1, it becomes $aS_3 + bS_2 + cS_1 + dS_0 + eS_{-1} = 0$
➔which can then be used to find the value of S_3

Substitutions

➔when given a substitution like y = x ± n or $y = x^n$
➔rearrange the substitution so that x is the subject and then substitute into the equation
➔this gives you an equation with roots x = α ± n or $x = \sqrt[n]{\alpha}$ depending on the substitution

- Sum of roots
➔for a substitution of $x = y^2$, all of the roots are squared
➔thus S_1 of the new equation is equivalent to S_2 of the original equation
➔for a substitution of $x = y^3$, all of the roots are cubed
➔thus S_1 of the new equation is equivalent to S_3 of the original equation

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Rational functions and graphs

Asymptotes

- Vertical asymptotes
➔equating denominator to 0 gives the vertical asymptote
➔for example for y = $\frac{x^2+2x+3}{x-3}$, x-3 = 0, gives us the vertical asymptote x = 3
➔the curve approaches +∞ on one side and −∞ on the other side
➔a curve can never cross a vertical asymptote

- Double vertical asymptote
➔if there is a double vertical asymptote at x = a
➔the curve approaches +∞ or -∞ on both sides of the double asymptote x = a

- Horizontal asymptotes
➔if denominator and numerator have the same power of x
➔divide the numerator and denominator with the highest power of x
➔as x→∞, the expression gives the horizontal asymptote
➔for example for y = $\frac{x^2+2x+3}{2x^2-x-3}$
➔when u divide by x^2 and as x→∞ , y = $\frac{1}{2}$ is the horizontal asymptote
➔if the power of the numerator is less than the power of the denominator
➔the line y = 0 is a horizontal asymptote
➔for example, y = $\frac{2}{x^2+2x+5}$ has horizontal asymptote y = 0
➔a curve can cross a horizontal asymptote

- Oblique asymptotes
➔if the power of the numerator is greater than the power of the denominator
➔use long division to get the quotient which is the oblique asymptote
➔for example, y = $\frac{x^2+2x+3}{x-3}$, use long division to get y = x + 5 +$\frac{18}{x-3}$
➔the oblique asymptote will be y = x + 5
➔a curve can cross a oblique asymptote

Intercepts

- X-intercepts
➔replacing y with 0 gives the x intercepts

- Y-intercepts
➔replacing x with 0 gives the y intercepts

- Horizontal/oblique asymptote intercepts
➔in the equation replace y with the horizontal/oblique asymptote and solve for x
➔Substitute the x-value into the equation to determine the y-value of the intercept

Turning points

➔differentiate the equation and equate to 0
➔$\frac{dy}{dx}$=0
➔then substitute these x-values into the original equation to get the y-values

Curve sketching

➔find and plot the vertical, horizontal, and oblique asymptotes
➔find and plot the x and y intercepts, and the horizontal asymptote intercept
➔behaviour of curve on either side of asymptotes
➔find and plot the coordinates of the turning points
➔Use these points and lines to sketch the graph

Curve range

➔to get set of values taken by the function, rearrange the equation so all terms are on one side
➔use $b^2 - 4ac \geq 0$ to determine the values of y when there are real solutions
➔use $b^2 - 4ac < 0$ to determine the values of y when there are no real solutions

Solving inequalities

➔find the find the set of values of x for $|\frac{2}{(x+2)(x-3)}| > 3$
➔removing the modulus sign gives you two inequalities
➔$\frac{2}{(x+2)(x-3)} > 3$ and $\frac{2}{(x+2)(x-3)} >- 3$
➔solve each inequality to find critical x-values
➔use the graph along with critical points to give the set of values of x

Graph relationships

- $y = |f(x)|$
➔the curve below the x-axis (y < 0) gets reflected above the x-axis

- $y = f(|x|)$
➔The curve to the left of the y-axis (x < 0) is reflected to the right of the y-axis

- $y = \frac{1}{f(x)}$

➔x-intercepts become vertical asymptotes
➔vertical asymptotes become x-intercepts
➔horizontal asymptotes (y = a) are replaced by their reciprocal ($y = \frac{1}{a}$) where $a \neq 0$
➔oblique asymptote becomes horizontal asymptote $y = 0$
➔maximum point (x,y) becomes minimum point (x,$\frac{1}{y}$) and vice versa
➔increasing graph becomes decreasing graph and vice versa

- $y^2 = f(x)$

➔curve is symmetrical about x-axis
➔sketch $y = \sqrt{f(x)}$ and then reflect it in the x-axis to sketch $y = -\sqrt{f(x)}$
➔turning point (x,y) becomes (x,$\sqrt{b}$) and (x,-$\sqrt{b}$)
➔horizontal asymptote y = a becomes y =$\sqrt{a}$ and y = -$\sqrt{a}$
➔graph of $y = f(x)$ and $y^2 = f(x)$ intersect at y = 1 and y = 0
➔for 0<y<1 , the graph of $y^2 = f(x)$ is above $y = f(x)$
➔for y>1 , the graph of $y^2 = f(x)$ is below $y = f(x)$

Summation of series

→ $\sum_{r=1}^{n} 1 = n$

→ $\sum_{r=1}^{n} r = \frac{1}{2}n(n + 1)$

→ $\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n + 1)(2n + 1)$

→ $\sum_{r=1}^{n} r^3 = \frac{1}{4}n^2(n + 1)^2$

Summation rules

- Addition rule

→ $\sum_{r=1}^{n} r^2 + r = \sum_{r=1}^{n} r^2 + \sum_{r=1}^{n} r$

- Multiple rule

→ $\sum_{r=1}^{n} br^2 = b\sum_{r=1}^{n} r^2$

Method of differences

→use the method of differences to find $\sum_{r=1}^{N} f(r) - f(r + 1)$

→write the first few terms of the sum to identify the cancellation pattern

→ $\sum_{r=1}^{N} f(r) - f(r + 1) = f(1) - f(2) + f(2) - f(3) + ... + f(N) - F(N + 1)$

→simplify by cancelling out all the terms

→ $\sum_{r=1}^{N} f(r) - f(r + 1) = f(1) - f(N + 1)$

Sum to infinity

→as $N \to \infty$, anything with N in its denominator approaches 0

→ $\frac{1}{N} \to 0$

→thus the remaining equation is the sum to infinity $\sum_{r=1}^{\infty} U_r$

Matrices

➔ a $m \times n$ matrix has m rows and n columns

Matrix types

- Zero Matrix

➔a matrix containing only zeros

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

- Identity / Unit matrix

➔a square matrix where all the diagonal elements are 1 and all other elements are 0

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

➔any matrix multiplied by identity/unit matrix remains the same

➔AI = A , IA = A

Matrix operations

- Adding two matrices

➔to add two matrices A and B, add corresponding elements of A and B.

➔condition : matrices A and B must be of the same size

➔A + B = B + A

$$\begin{pmatrix} 2 & 3 \\ 3 & 1 \end{pmatrix} + \begin{pmatrix} 4 & 2 \\ 3 & 2 \end{pmatrix} = \begin{pmatrix} 6 & 5 \\ 6 & 3 \end{pmatrix}$$

- Subtracting two matrices

➔to add two matrices A and B, add corresponding elements of A and B.

➔condition : matrices A and B must be of the same size

➔A - B = -B + A

$$\begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix} - \begin{pmatrix} 3 & 0 \\ 3 & 2 \end{pmatrix} = \begin{pmatrix} -1 & 3 \\ -2 & 2 \end{pmatrix}$$

- Multiplying matrices

➔to multiply matrix A by matrix B, multiply each element in the rows of A by the corresponding

➔element in the columns of B, then sum the products

➔condition : no of row of matrix A = no of columns of matrix B

➔AB ≠ BA

➔A(BC) = (AB)C

$$\begin{pmatrix} 2 & 3 & 2 \\ 0 & 4 & 1 \end{pmatrix} \begin{pmatrix} 1 & 3 \\ 2 & 2 \\ 1 & 4 \end{pmatrix} = \begin{pmatrix} 10 & 20 \\ 9 & 12 \end{pmatrix}$$

- Matrix powers

➔for a matrix raised to the power of n, you will multiply the matrix by itself n times

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^3 = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Determinant of a matrix

➔a matrix with determinant = 0 is called a singular matrix

➔a matrix with determinant ≠ 0 is called a non-singular matrix

➔for a nxn matrix M, $det|kM|$ = $k^n det|M|$

- Determinant of a 2x2 matrix

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = (ad - bc)$$

- Inverse of a 2x2 matrix

➔inverse of a matrix M = $\frac{1}{det|M|}$(adjugate matrix) , $det|M| \neq 0$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

- Determinant of a 3x3 matrix

$$\det \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = a \left(\det \begin{pmatrix} e & f \\ h & i \end{pmatrix} \right) - b \left(\det \begin{pmatrix} d & f \\ g & i \end{pmatrix} \right) + a \left(\det \begin{pmatrix} d & e \\ g & h \end{pmatrix} \right)$$

$$= a(ei - fh) - b(di - fg) + c(dh - eg)$$

- Inverse of a 3x3 matrix

➔inverse of a matrix M = $\frac{1}{det|M|}$(adjugate matrix) , $det|M| \neq 0$

➔for the adjugate matrix replace every element with its respective minor matrix

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \Rightarrow \begin{pmatrix} A & B & C \\ D & E & F \\ G & H & I \end{pmatrix}$$

➔where A = (ei - fh), B = (di - fg), C = (dh - eg), D = (bi - ch), E = (ai - cg)... I = (ae - bd)

➔then transpose the matrix (swap each row and column)

$$\begin{pmatrix} A & B & C \\ D & E & F \\ G & H & I \end{pmatrix} \Rightarrow \begin{pmatrix} A & D & D \\ B & E & H \\ C & F & I \end{pmatrix}$$

➔then form the adjugate matrix

$$adj(M) = \begin{pmatrix} A & -D & D \\ -B & E & -H \\ C & -F & I \end{pmatrix}$$

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}^{-1} = \frac{1}{\det|M|}\begin{pmatrix} A & -D & D \\ -B & E & -H \\ C & -F & I \end{pmatrix}$$

- Matrix properties

➔multiplying a matrix with its inverse will give you the identity matrix

➔MM^{-1} = I

➔$(AB)^{-1}$= $B^{-1}A^{-1}$

Geometric transformations

➔the matrix product AB represents the transformation of applying matrix B first, then matrix A

➔the matrix A^{-1} is the inverse transformation of A, undoing the effect of A

- Change in Area from transformation matrix

➔if a shape with area A is transformed by matrix M

➔Image area = shape area * $det|M|$

➔a singular matrix maps points to a straight line, resulting in zero area

- Stretch

➔stretch parallel to x-axis with scale factor k

➔y-axis is an invariant line and the origin is the invariant point

$$\begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}$$

➔stretch parallel to y-axis with scale factor k

➔x-axis is an invariant line and the origin is the invariant point

$$\begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix}$$

- Enlargement

➔enlargement with origin as the center of scale factor k

➔origin is the invariant point

$$\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$$

- Reflection

➔reflection in the x-axis

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

➔reflection in the y-axis

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

➔reflection in the line y = x

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

➔reflection in the line y = -x

$$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

- Rotation

➔Rotation of θ anticlockwise about the origin

➔origin is the invariant point

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

- Shear

➔shear parallel to x-axis and scale factor k

➔y-axis is the invariant line and the origin is the invariant point

$$\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$$

➔shear parallel to y-axis and scale factor k

➔x-axis is the invariant line and the origin is the invariant point

$$\begin{pmatrix} 1 & 0 \\ k & 1 \end{pmatrix}$$

Invariant Points & Lines

- Invariant point

➔a point that stays the same under a given transformation

➔if M * P = P

➔where M is a transformation matrix and p is a point

➔then p would be an invariant point

- Finding the invariant point

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

➔first multiply the point and the transformation matrix and equate to the point

➔this will give you two simultaneous equations

➔$ax + by = x$

➔$cx + dy = y$

➔solve the simultaneous equations to get values for x and y

- Invariant line

➔a line where points may move but are mapped onto the same line under the transformation

- Finding the equation of the invariant line through the origin

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} X \\ Y \end{pmatrix}$$

➔y = mx , Y = mX

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ mx \end{pmatrix} = \begin{pmatrix} X \\ mX \end{pmatrix}$$

➔simplify this to get two equations

➔ax + bmx = X

➔cx + dmx = mX

➔divide equation 1 by equation 2 and simplify

➔$\frac{ax+bmx}{cx+dmx}=\frac{X}{mX}$

➔this will simplify into a quadratic equation which u can solve for the value of m (gradient)

➔then substitute this value of m into y = mx to get your equation of the invariant line

Polar Coordinates

Polar and Cartesian

➔r is the distance from the pole
➔ θ is the anticlockwise measured from the line θ = 0 (measured in radians)
➔x = rcos(θ)
➔y = rsin(θ)
➔r = $\sqrt{x^2 + y^2}$
➔ θ = tan($\frac{x}{y}$)

- Cartesian to polar
➔to convert cartesian equations (x,y) to polar equations (r,θ)
➔turn all x's and y's into their respective polar counterparts
➔sometimes it's necessary to group values or manipulate the given equation

- Polar to cartesian
➔to convert polar equation (r,θ) to cartesian equation (x,y)
➔turn all r's and sin(θ)/cos(θ)'s into their respective polar counterparts
➔sometimes it's necessary to group values or manipulate the given equation

Sketching Polar curves

➔to sketch a polar curve, substitute values of θ within the given range in the polar equation
➔now using these values of r sketch the polar curve
➔the convention r>0 will be used so ignore any negative values of r

- Symmetry
➔the curve is symmetric about the x-axis if f(-θ) = f(θ)
➔if r is a function of cosθ only, then it is symmetrical about the x-axis
➔the curve is symmetric about the y axis if f(π-θ) = f(θ)
➔if r is a function of sinθ only, then it is symmetrical about the y-axis
➔the curve is symmetric about the if replacing r with -r produces an equivalent equation

- Circles
➔r = a will give a circle or radius a centre on the pole
➔r = 2asinθ will give a circle of radius a, centre on the line θ = $\frac{1}{2}$π at r = a
➔r = 2acosθ will give a circle of radius a, centre on the initial line at r = a

- Limacons
➔r = a $\pm$ bsinθ and r = a $\pm$ bcosθ
➔if the ratio of a and b is equal to 1, it will be a cardioid
➔if the ratio of a and b is smaller than 1, it will be a limacon with a inner loop
➔if the ratio of a and b is bigger than 1 and smaller than 2, it will be a dimpled limacon
➔if the ratio of a and b is bigger than or equal to 2, it will be a convex limacon

- **Rose curves**
➔r = a$sin(n\theta)$ and r = a$cos(n\theta)$
➔if n is an odd number, the rose curve will have n petals
➔if n is an even number, the rose curve will have 2n petals

- **Spirals**
➔r = aθ gives a archimedean spiral
➔r = a$\sqrt{\theta}$ gives a parabolic spiral
➔r = $\frac{a}{\theta}$ gives a hyperbolic curve

Maximum and minimum values

- **Finding r_{max}**
➔$\frac{dr}{d\theta} = 0$
➔this gives the maximum distance from the pole

- **Finding y_{max} / y_{min}**
➔ $\frac{dx}{d\theta} = 0$
➔substitute this θ to get the maximum/minimum distance from the line θ = $\frac{1}{2}\pi$

- **Finding x_{max} / x_{min}**
➔ $\frac{dy}{d\theta} = 0$
➔substitute this θ to get the maximum/minimum distance from the initial line

Area of Polar curves

- **Area of a polar curve**
➔Area = $\frac{1}{2}\int_a^b r^2 d\theta$

- **Area between two polar curves**
➔Area = $\frac{1}{2}\int_a^b r_1^2 d\theta$ - $\frac{1}{2}\int_a^b r_2^2 d\theta$
➔Area = $\frac{1}{2}\int_a^b r_1^2 - r_2^2 d\theta$
➔where b and a are the intersection points

Vectors

Vector product

➔ A x B = |A||B|sinθ$\hat{n}$ = (a_2b_3 - a_3b_2)i - (a_1b_3 - a_3b_1)j + (a_1b_2 - a_2b_1)k
➔ the cross product of Vector A and Vector B is a unit vector perpendicular to both vectors

- Finding cross product
➔The cross product of two vectors can be found by the determinant of a 3x3 matrix

$$a \times b = \det \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix} = (a_2b_3 - a_3b_2)\hat{i} - (a_1b_3 - a_3b_1)\hat{j} + (a_1b_2 - a_2b_1)\hat{k}$$

- Area of Parallelogram
➔ |a × b|
➔magnitude of the cross product

- Area of triangle
➔$\frac{1}{2}$|a × b|

Equation of a plane

- Parametric form
➔ r = a + λb + μc
➔ a is a known point on the plane
➔ b and c are direction vectors

- Scalar Product form
➔$\hat{r} \cdot (n)$ = d
➔n is the normal vector of the plane
➔d is the distance from origin

- Cartesian form
➔ax + by + cz = d
➔(a,b,c) is the normal vector

- Parametric to scalar product
➔direction vector n = b x c
➔d = a·n
➔substitute these into scalar product form equation

- Scalar product to Cartesian
➔expand the scalar product equation by dot product of $\hat{r} \cdot (n)$ to get the cartesian equation

Finding equation of a plane

- Given 2 lines on a plane

➔substitute the direction vectors of the lines into the plane equation
➔use any lines known point for ur a

- Given 1 line on a plane and 1 parallel line

➔ substitute the direction vectors of both lines into the plane equation
➔use the line on the planes point for your a

- Given a point and a line on a plane

➔use the point on the line (OA) and the point on the plane (OB) to get a direction vector (AB)
➔AB = OB - OA
➔substitute AB and the lines direction vector into the plane equation.
➔use either OA or OB for ur a

- Given 3 points on the plane

➔use your points (OA, OB, OC) to make two direction vectors.
➔AB = OB - OA and AC = OC - OA
➔substitute AB and AC into the plane equation
➔use any of the three points for your a

Line and plane

- Line lies in the plane

➔dot product of the normal vector and the lines direction vector should be 0 as cos(90) = 0
➔n·d ≠ 0
➔now find the coordinate of the line at that point using λ = n (n can be any number)
➔substitute that coordinate into the cartesian equation of the plane
➔the equation must become true to show that the point on the line lies on the plane

- Line parallel to the plane

➔dot product of the normal vector and the lines direction vector should be 0 as cos(90) = 0
➔n·d = 0
➔there are no intersection points between the line and plane
➔now find the coordinate of the line at that point using λ = n (n can be any number)
➔substitute that coordinate into the cartesian equation of the plane
➔the equation must become false to show that the equation does not intersect

- Line intersects the plane

➔dot product of normal vector and lines direction vector should not 0
➔n·d ≠ 0

- Point of intersection between line and plane

➔Find general point of line and substitute into cartesian equation of plane
➔solve the equation to get a value for λ
➔now use this value of λ in your general point of line to get point of intersection

Point and plane

- Foot of the perpendicular from a point to a plane

➔use the normal vector of the plane and the point to create a line equation
➔Find general point of line and substitute into cartesian equation of plane
➔solve the equation to get a value for λ
➔now use this value of λ in your general point of line to get the foot of perpendicular.

- Perpendicular distance between a point and a plane

➔distance = $\frac{|ax+by+cz-d|}{\sqrt{a^2+b^2+c^2}}$
➔(a,b,c) is the normal vector of the plane
➔d is the distance of the plane from the origin (scalar product form)
➔(x,y,z) are coordinates of the point

Angles

- Angle between a line and a plane

➔rearranging the dot product of the normal vector of the plane and the direction vector gives α
➔$d \cdot n$ = |d|n||cosα becomes $\alpha = cos^{-1}\left(\frac{d \cdot n}{|d||n|}\right)$
➔d and n are the direction vector of the line and the normal vector of the plane respectively
➔if α < 90 then θ = 90 - α
➔if α > 90 then θ = α - 90
➔if α = 90 then θ = 0
➔if α = 0 then θ = 90

- Angle between two planes

➔rearranging the dot product of the normal vectors gives you the angle between the two planes
➔$n1 \cdot n2$ = |n1|n2||cosα becomes $\alpha = cos^{-1}\left(\frac{n1 \cdot n1}{|n1||n2|}\right)$
➔n1 and n2 are the normal vectors of the planes
➔if the value inside cos^{-1} is negative then your acute angle will be (90-α)

Line of intersection of two planes

➔cross product of the normal vectors will give us the direction vector for the line of intersection
➔equating a variable to zero in the cartesian equations will give us values for the other variables
➔this common point between the planes will be our point for the line
➔substitute these to get the line of intersection

Perpendicular distance between parallel planes

➔take a point on one of the planes and then use the distance between a point and a plane

➔distance = $\frac{|ax+by+cz-d|}{\sqrt{a^2+b^2+c^2}}$

➔substitute the values into the formula above to get the perpendicular distance

Skew lines

- Shortest distance between two skew lines

➔distance = $\frac{|(a1-a2)\cdot(d1\times d2)|}{|(d1\times d2)|}$

➔a_1 and a_2 are the points of the skew lines

➔d_1 and d_2 are the direction vectors of the skew lines

➔distance = $\frac{|a\cdot n|}{|n|}$

➔a is a direction vector between the two skew lines

➔n is the perpendicular vector between the two skew lines

- Common perpendicular of two skew lines

➔common perpendicular direction vector = d1 x d2

➔where d1 and d2 are the direction vectors of the skew lines

➔Find the general point on both skew lines (OP , OQ)

➔PQ = OQ - OP

➔dot product of PQ and direction vectors of skew lines should be zero as cos(90) = 0

➔PQ · d1 = 0 will give us a value for λ

➔PQ · d2 = 0 will give us a value for μ

➔Use these values of λ *and* μ in either one of the general points (OP or OQ) to get the point

➔this gives the common perpendicular between these two skew lines

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Proof by induction

General steps

- Statement of the proposition

➔Prove that P_n is true for $n \geq x$

- Basis case

➔Prove P_n is true for $n = x$

➔x is the smallest value that n can be

- Inductive Hypothesis

➔Assume P_n is true for $n = k$

- Inductive Step

➔Use the inductive Hypothesis to prove the statement P_n for $n = k+1$

- Completion of Proof

➔Since P_x is true and P_k leads to P_{k+1} , by mathematical induction, P(n) is true for $n \geq x$

Proof types

- Summation Proofs

➔Prove P_n , a summation formula, for $n \geq 1$

➔prove it for $n = 1$

➔Assume it is true for $n = k$

➔Prove when $n = k + 1$ by equating P_{k+1} to the sum of P_k and the (k+1)th term of the formula

➔$P_{k+1} = P_k + f(k+1)$

➔Since P_1 is true and P_k leads to P_{k+1} , by mathematical induction, P_n is true for $n \geq 1$

- Recurrence relation Proofs

➔Prove P_n , the general solution to a recurrence relation, for $n \geq 1$

➔prove it for $n = 1$

➔Assume it is true for $n = k$

➔Prove when $n = k + 1$ by substituting P_k into the recurrence relation and showing it equals P_{k+1}

➔$P_{k+1} = aP_k + b$ (depends on the recurrence relation)

➔Since P_1 is true and P_k leads to P_{k+1} , by mathematical induction, P_n is true for $n \geq 1$

- Inequality Proofs

➔Prove P_n , an inequality, for $n \geq 1$

➔prove it for $n = 1$

➔Assume it is true for $n = k$

➔Prove when $n = k + 1$ by manipulating the inequality P_k to show that P_{k+1} holds

➔Since P_1 is true and P_k leads to P_{k+1} , by mathematical induction, P_n is true for $n \geq 1$

- Matrix Proofs

➔Prove P_n , the n-th power of a matrix, for n ≥ 1
➔prove it for n = 1
➔Assume it is true for n = k
➔Prove when n = k+1 by equating P_{k+1} to P_k times P_1
➔$P_{k+1} = P_k * P_1$
➔Since P_1 is true and P_k leads to P_{k+1} , by mathematical induction, P_n is true for n ≥ 1

- Divisibility Proofs

➔ Prove P_n is divisible by the number a for n ≥ 1
➔prove it for n = 1
➔Assume it is true for n = k
➔To prove when n = k+1. The difference between P_{k+1} and P_k should be divisible by a
➔$(P_{k+1} - P_k)$ should be divisible by a
➔Since P_1 is true and P_k leads to P_{k+1} , by mathematical induction, P_n is true for n ≥ 1

- Derivative Proofs

➔Prove P_n , the n-th derivative of a function, for n ≥ 1
➔prove it for n = 1
➔Assume it is true for n = k
➔Prove when n = k+1 by equating P_{k+1} to the differentiation of P_k
➔$P_{k+1} = \frac{d}{dx}(P_k)$
➔Since P_1 is true and P_k leads to P_{k+1} , by mathematical induction, P_n is true for n ≥ 1

- Factorial Proofs

➔ Prove P_n , a statement involving factorials, for n ≥ 1
➔prove it for n = 1
➔Assume it is true for n = k
➔Prove when n = k+1 by expressing P_{k+1} in terms of P_k using the factorial property
➔$P_{k+1} = P_k * (k+1)$
➔Since P_1 is true and P_k leads to P_{k+1} , by mathematical induction, P_n is true for n ≥ 1

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A Note from Mojza

These notes for Further Mathematics (9231) have been prepared by Team Mojza, covering the content for AS Level 2025-27 syllabus. The content of these notes has been prepared with utmost care. We apologise for any issues overlooked; factual, grammatical or otherwise. We hope that you benefit from these and find them useful towards achieving your goals for your Cambridge examinations.

If you find any issues within these notes or have any feedback, please contact us at support@mojza.org.

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9231

AS Level

FURTHER MATHS

PAPER 1 - NOTES

These notes are made to encompass the complete syllabus for 9231 from 2025 to 2027, with great attention and care for every topic. All information is curated in a simple, clear, and concise manner. The aim is to aid students and make learning easier in preparation for their exams. Team Mojza makes every effort to error-check all the content; if you find any discrepancies, please reach out to us at *support@mojza.org*.