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Mathematics Notes (P3)

9709

BY TEAM MOJZA

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CONTENTS

Equations and Inequalities

Modulus Functions

- The modulus function makes any 'input' positive
- |x| = x if x ≥ 0 |f(x)| = f(x) if f(x) ≥ 0
- |x| = -x if x < 0 |f(x)| = -f(x) if f(x) < 0
- For example: |5| = 5 and |-5| = 5
- Sometimes called absolute value

Properties of modulus functions

- $|a \times b| = |a| \times |b|$
- $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$
- $\left|x^2\right| = |x|^2 = x^2$
- $|x| = |a| \Leftrightarrow x^2 = a^2$
- $\sqrt{x^2} = |x|$
- For $|f(x)| = |g(x)|$ the two possible equations are f(x)=g(x) and f(x)=-g(x)

Sketching graph of modulus function y=$|ax + p|$+q

- There will be a vertex at the point (-*p*, *q*)
- There could be 0, 1 or 2 roots. This depends on the location of the vertex and the orientation of the graph

Sketching graph of modulus function y=$|f(x)|$

- Pencil in the graph of y=f(x)
- Reflect anything below the x-axis in x-axis to get y=$|f(x)|$

Remainder and Factor Theorem

A polynomial is an algebraic expression consisting of a finite number of terms, with non-negative integer indices only

Factor Theorem

- For a polynomial f(x) the factor theorem states that:

  If f(p) = 0, then (x - p) is a factor of f(x)
- If (x - p) is a factor of f(x), then f(p) = 0
- The factor theorem is a very useful result about polynomials

Remainder Theorem

- The remainder theorem states that when the polynomial f(x) is divided by (x - a) the remainder is f(a)

  $$f(x) = (x - a)Q(x) + f(a)$$
- Q(x) would be the result (at the top) of the division (the quotient)
- f(a) would be the remainder (at the bottom)
- (x - a) is called the divisor
- In the case when f(a) = 0, f(x) = (x - a)Q(x) and hence (x - a) is a factor of f(x)– the factor theorem

Partial Fractions and Binomial Expansion

Partial Fractions:

Solving partial fractions is obtaining individual fractions and operations from a given answer.

STEP 1-

Check if the fraction is proper or improper
If the highest power of the variable in denominator is greater than that of numerator - **Proper**
If the highest power in denominator is equal to or less than the numerator - **Improper**

Converting improper to proper-
Using a long division method to bring in proper fraction form.

STEP 2 -

Once the fraction is proper, figure its type.
Place unknown variables as a numerator for each bracket.

LINEAR AND DISTINCT $\frac{1}{(x+1)(x+2)}$	The brackets in the denominator have X to power 1 and are distinct.	$\frac{A}{(x+1)} + \frac{B}{(x+2)}$
LINEAR AND REPEATED $\frac{1}{(x+1)(x+2)^2}$	The brackets in denominator have X power 1 but are repeating themselves	$\frac{A}{(x+1)} + \frac{B}{(x+2)} + \frac{C}{(x+2)^2}$
QUADRATIC $\frac{1}{(x+1)(x^2+2)}$	The brackets in the denominator have X square. Put a linear equation instead of a variable in the numerator.	$\frac{A}{(x+1)} + \frac{Bx+C}{(x^2+2)}$

STEP 3-

Take LCM for the unknown variable form.
And equate the numerator to the one in question.
e.g:

$$\frac{1}{(x+1)(x+2)} = \frac{A}{(x+1)} + \frac{B}{(x+2)}$$

$$1 = A(x + 2) + B(x + 1)$$

STEP 4-

Substitute values for X
(find values by equating brackets to 0, substitute 1 and 0 for quadratic form.) and solve for A and B.
Write the final answer in the form written in Step 2.

Binomial Expansion

$$(1 + ax)^n$$

Used only when:

- n is a negative number
- n is a fraction

Format:

- Must always be $(1 + ax)^n$
- If the bracket is not in format- first digit not 1, convert it to correct form by taking out common.

Once in correct form, Apply the formula using calculator

$$1 + (n)(x) + \frac{(n)(n-1)(x^2)}{2!} + \frac{(n)(n-1)(n-2)(x^3)}{3!} + \frac{(n)(n-1)(n-2)(n-3)(x^4)}{4!} ....$$

Logarithmic and Exponential Functions

Logarithmic Functions

- The logarithm of a number n to a given base a, is the power x to which the base a must be raised to obtain n
- $x = \log_a n \Leftrightarrow a^x = n$ where $a > 0$, $a \neq 1$ and $n > 0$
- $x = \log_a n$ is called the logarithmic form
- $a^x = n$ is the exponential form.
- $\log_{10} x$ is called common logarithm written as lg
- ln denotes $\log_e x$ called natural or Napierian logarithm where $e \approx 2.71828 \ldots$ is an irrational number

Laws Of Logarithms

- $\log_a xy = \log_a x + \log_a y$
- $\log_a \frac{x}{y} = \log_a x - \log_a y$
- $\log_a x^k = k \log_a x$
- $\log_a a = 1$
- $\log_a 1 = 0$

Exponential Functions

- Try to express both sides of the equation to a common base in order to solve it. If unable to express both sides to a common base then take log to the same base on both sides
- Look out for disguised quadratics in equations having exponential terms with powers in the ratio 1: 2

Inequalities

- When taking logs on both sides, if the base is greater than 1 then the sign of inequality does not change, for base less than 1 it would flip
- In case of dividing on both sides of an inequality by $\log x$ where $0 < x < 1$, the sign would flip

Linearizing equations

- ➔ $y = kx^n \Leftrightarrow \log y = n \log x + \log k$ where k and n are constants
- ➔ Comparing with $Y = mX + c$, gives us $Y = \log y$ and $X = \log x$, with n as the gradient of the straight line and $\log k$ as the y-intercept

- ➔ $y = ka^x \Leftrightarrow \log y = (\log a)x + \log k$ where k and n are constants
- ➔ Comparing with $Y = mX + c$, gives us $Y = \log y$ and $X = \log x$, with n as the gradient of the straight line and $\log k$ as the y-intercept

Trigonometry

R-method questions

- Examiner usually gives an equation and asks the candidate to express it in the form of Rsin (θ − α) or Rcos (θ − α).
- To find R you will apply R= $\sqrt{a^2 + b^2}$
- To find alpha, you will apply, $tan\theta = \frac{sin\theta}{cos\theta}$

Identities

$\frac{1}{sin\theta} = cosec\theta$	$sin^2\theta + cos^2\theta = 1$
$\frac{1}{cos\theta} = sec\theta$	$1 + tan^2\theta = sec^2\theta$
$\frac{1}{tan\theta} = cot\theta$	$1 + cot^2\theta = cosec^2\theta$
$tan\theta = \frac{sin\theta}{cos\theta}$	$cot\theta = \frac{cos\theta}{sin\theta}$

sin(180-θ) = sinθ	sin(90-θ)= cosθ
cos(180-θ)= -cosθ	cos(90-θ)= sinθ
tan(180-θ) = -tanθ	tan(90-θ)= $\frac{1}{tan\theta}$

Double Angle Identities

- sin2A = 2sinAcosA
- cos2A = $cos^2A - sin^2A$ **OR** $2cos^2A - 1$ **OR** $1 - 2sin^2A$
- tanA = 2 tan A / (1 - tan^2A)

Compound Angle Identities

- sin (A+B) = sinAcosB + cosAsinB [Sign remains the same]
- cos (A+B) = cosAcosB - sinAsinB [Opposite sign is applied]
- tan(A+B) = (tanA+tanB)/(1-tanAtanB) [Same sign in the numerator but opposite sign in the denominator]

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Identity Proving

- ➔ When proving identities, keep an eye on the angle to analyse whether double or compound identities need to be applied
- ➔ Identity terms cannot change sides.
- ➔ Always solve one side and prove it equal to the other side.

Differentiation

- Gradient of a tangent

Operators

-Power

$(\square)^n \rightarrow n(\square)^{n-1} . \square`$

- The box represents the equation.
- To differentiate, multiply the original power with the original equation and subtract the original power with 1.
- Then multiply the whole thing by the derivative of the equation in the box.

Example

$y = (2x+5)^3$

$\frac{dy}{dx} = 3(2x+5) . (2)$

$\frac{dy}{dx} = 6(2x+5)^2$

-Trigonometry

$\sin\square \rightarrow \cos\square . \square`$
$\cos\square \rightarrow -\sin\square . \square`$
$\tan\square \rightarrow \sec^2\square . \square`$

- Sine is converted into cosine and the derivative of the box is multiplied by it.
- Cosine is converted into negative sine and the derivative of the box is multiplied by it.
- Tangent is converted into secant squared and the derivative of the box is multiplied by it.

Example

$y = \sin 4x$

$\frac{dy}{dx} = \cos 4x. (4)$

$\frac{dy}{dx} = 4\cos 4x$

-Exponential

$e^{\square} \rightarrow e^{\square} . \square$

- Exponential remains the same and it is multiplied by the derivative of the power.

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Example

$y = e^{3x}$

$\frac{dy}{dx} = e^{3x} . (3)$

$\frac{dy}{dx} = 3e^{3x}$

-Logarithmic

$\ln(□) = \frac{1}{\ln □} . □`$

➔ Reciprocal of the box is taken and multiplied by derivative of box

Example

$y = \ln(2x+3)$

$\frac{dy}{dx} = \frac{1}{2x+3} . (2)$

$\frac{dy}{dx} = \frac{2}{2x+3}$

-Inverse Tan

$\tan^{-1}□ = \frac{1}{1+□^2} . □`$

➔ To differentiate, a fraction will be formed with 1+ $□^2$ in the denominator and 1 in the numerator.

Example

$y = \tan^{-1}(4x+6)$

$\frac{dy}{dx} = \frac{1}{1+(4x+6)^2} . (4)$

$\frac{dy}{dx} = \frac{4}{1+(4x+6)^2}$

-Multiple Operators

➔ For multiple operators, differentiate the outer operator first.

Example

$y = \cos^2 6x$

$y = (\cos 6x)^2$

$\frac{dy}{dx} = 2(\cos 6x)^1 . (-\sin 6x . 6)$

$\frac{dy}{dx} = -12\cos 6x \sin 6x$

Product and Quotient Rule

➔ Product and quotient rule is applied when two variable terms are being multiplied/divided.

Product Rule	Quotient Rule
(uv)\`= uv\`+vu\`	$(\frac{u}{v})` = \frac{uv`-vu`}{v^2}$

Tangent, Normal and Stationary Point

The tangent is a straight line that touches the curve at just 1 point. The normal is a straight line that is perpendicular (at a 90 degree angle) to the tangent.

The stationary point of the curve is where the derivative of the curve is equal to zero that is dy/dx = 0

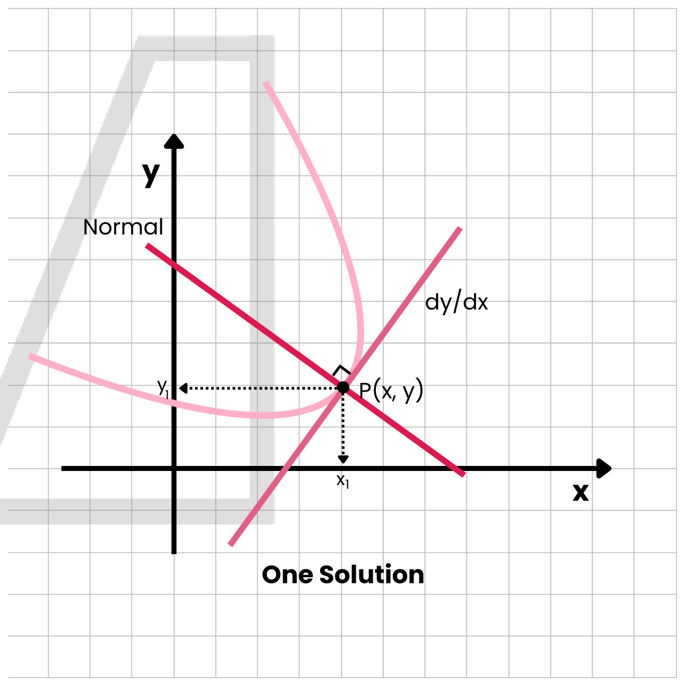
Gradient of tangent = $\frac{dy}{dx}$

On the right, figure P is the point of tangency and (x,y) are the coordinates of the point of tangency.

If $\frac{dy}{dx}$ = ¾

Then m_n = - 4/3

Where m_n is the gradient of the normal



Example

Given the equation of the curve , $x^2 + 2x$, at the point (2,1) , find the equations of the tangent and normal

Solution

$\frac{dy}{dx}$=$2x + 2$

At x = 2

$\frac{dy}{dx}$ = 2(2) + 2 = 6

$\frac{dy}{dx}$= 6 m_n = -⅙

Equation of tangent
(y-1) = 6 (x-2)
y-1 = 6x -12
$y = 6x - 11$

Equation of normal
(y-1) = -⅙ (x-2)
6y-6 = -x +2
$6y + x = 8$

Parametric Equations

- Parametric differentiation is done when "x" and "y" are given as a function of a 3rd variable.
- The 3rd variable is called the parameter and the 2 equations are called parametric equations
- To sum it, two equations connecting 3 variables are called parametric equations.

The gradient of parametric equations is $\frac{dy}{dt}$ / $\frac{dx}{dt}$ where t is the 3rd variable.

Examples

1. Given that x=$sin\ t$ y=$cos\ t$

$\frac{dx}{dt}$= cos t $\frac{dy}{dt}$= -sin t

$\frac{dy}{dx}$= $\frac{dy}{dt}$/ $\frac{dx}{dt}$
-sin t / cos t = -tan t

2. x= $t^2 + t$ y= $2t - 1$

$\frac{dx}{dt}$ = $2t + 1$ $\frac{dy}{dt}$=2

$\frac{dy}{dx}$ =$\frac{dy}{dt}$ / $\frac{dx}{dt}$

$\frac{dx}{dt}$ = $2t + 1$ $\frac{dy}{dt}$=2

$\frac{dy}{dx} = \frac{dy}{dt} / \frac{dx}{dt}$

$\frac{dy}{dx}$ = $2 / 2t + 1$

Implicit Differentiation

$y = 9x^2 + 4x + 5$

The above mentioned equation is an explicit function because y is expressed in terms of x . However , that is not the case always

$5x^4 + 3xy + y^2 = 22$
In this case , y is not the subject of this equation hence its an implicit function

Generally , implicit means "the differential of y^n w.r.t to x "

The method for solving this is the same as we solve normal functions.The only addition is that the differential of y is multiplied by dy/dx in all implicit functions.

Example
y^4 = 5y^2

$\frac{dy}{dx}$ 4y^3= $\frac{dy}{dx}$ $10y$

Integration

Integration is the reverse of differentiation .The sign of integral is $\int$ and the integral operator is dx .

Once the function is integrated , we remove both $\int$ and dx and add the constant “c”.

Eg $\int x^5 dx$

$x^6/6 + c$

In general terms ,

$\int x^n dx$

= $x^{n+1}/n + 1 + c$

Integration of nonlinear functions

The nonlinear function needs to be expanded first and then integrated

Eg $(2 - 4x^2)^2$

$\int 4 - 16x^2 + 16x^4$

$4 - 16x^3/3 + 16x^5/5$ + c

Application of Integration

Area under the curve

1. Area between the curve and y axis
   On the right hand side of y axis
   Area = $\int_a^b x \; dy$

   On the left hand side of y axis

Area = $-\int_{a}^{b} x \ dy$

1. Area between the curve and x axis

   For above the x axis

   Area = $\int_{a}^{b} y \ dx$

   For below the x axis

   Area = $-\int_{a}^{b} y \ dx$

Example

Area under the curve

Find the area bounded by the curve y = x^2 and the x-axis from x = 0 to x = 2

$\int_0^2 y$ dx = $\int_0^2 x^2$dx

$\int_0^2 y$ dx = $\int_0^2 \frac{x^3}{3}$dx

$\int_0^2 y$ dx = $[(\frac{(2)^3}{3}) - (\frac{(0)^3}{3})]$

$\int_0^2 y$ dx = $[(\frac{8}{3}) - (\frac{0}{3})]$

$\int_0^2 y$ dx = $\frac{8}{3}$

Area = $\frac{8}{3}$

Area between two curves

find the area between the curves y = x^3 and y=x+2 from x = −1 to x = 1

y = x+2 is the upper curve

y = x^3 is the lower curve

$\int_{-1}^{1}(x+2)$ dx - $\int_{-1}^{1} x^3$ dx

$[\int_{-1}^{1}x \ dx + \int_{-1}^{1}2 \ dx] - \int_{-1}^{1} x^3$ dx

[$\int_{-1}^{1}\frac{x^2}{2}$ dx+$\int_{-1}^{1}$2x dx] - $\int_{-1}^{1}\frac{x^4}{4}$ dx

Apply limits

[{ ($\frac{(1)^2}{2}$) - ($\frac{(-1)^2}{2}$) } + { 2(1) - 2(-1) }] - [($\frac{(1)^4}{4}$)-($\frac{(-1)^4}{4}$)]

[($\frac{1}{2}$-$\frac{1}{2}$) + (2 + 2)] - [($\frac{1}{4}$-$\frac{1}{4}$)]

[0 + 4] - [0]

Area between curves = 4

Volume of Revolution

When the function f f(x) is rotated 360° about the x axis

Volume = $\pi \int_a^b y^2 \, dx$

And

When the function f f(x) is rotated 360° about the y axis

Volume = $\pi \int_a^b x^2 \, dy$

Integration of Exponential Functions

$\int e^{5x} dx$

e^{5x} /5 + c

Integration of ln functions

There is no integral of ln in Mathematics so instead integration by-parts is used for ln functions)

Integration of Trigonometric functions

1. $\int sin4x\, dx$

   $= \; -\, cos4x/4 + \; c$

2. $\int cos4x\, dx$

   $= \; sin4x/4 \; + \; c$

3. $\int sec^2 4x \;\; dx$

   $= \; tan4x/4 \; + \; c$

Integration of Arctan

$\frac{1}{1+\square^2} = \tan^{-1}\square \;/\square`$

- ➔ To integrate to arctan , there must be a fraction with $1+\square^2$ in the denominator and 1 in the numerator
- ➔ The box is placed inside arctan and then divided by the derivative of the box

Example

$y = \frac{1}{1+(2x+7)^2}$

$\int ydx = \int \frac{1}{1+(2x+7)^2} dx$

$\int ydx = \frac{1}{2}\tan^{-1}(2x \; + \; 7) + C$

Integration by U-substitution

- ➔ A substitution like u = $\square$ will be given
- ➔ differentiate the substitution to get $\frac{du}{dx} = \square`$
- ➔ rearrange to get dx = $\frac{du}{\square`}$
- ➔ now substitute every x value in the integral with u = $\square$ (you may need to re-arrange the substitution)
- ➔ then substitute dx with $\frac{du}{\square`}$
- ➔ now integrate the integral with respect to du
- ➔ change the upper and lower limits using the substitution u = $\square$ or substitute back the original x-terms if the integral was indefinite.

Example

$I = \int \frac{3x}{3x+1} dx$

u = 3x+1

$\frac{du}{dx}$ = 3

dx = $\frac{du}{3}$

I= $\int \frac{3x}{u}$. ($\frac{du}{3}$)

3x = u-1

I = $\int \frac{u-1}{u}$. ($\frac{du}{3}$)

I = $\int \frac{u-1}{3u}$du

I = $\int \frac{u-1}{3u}$du

I = $\frac{1}{3}$[$\int \frac{u}{u}$du - $\int \frac{1}{u}$du]

I = $\frac{1}{3}$[$\int$1du - $\int \frac{1}{u}$du]

I = $\frac{1}{3}$[u - ln(u)] + C

Substitute back x terms

u = 3x+1

I = $\frac{1}{3}$[(3x+1) - ln(3x+1)] + C

Integration by-parts

- The formula for integration by parts is:
  $\int$u$\frac{dv}{dx}$dx = uv - $\int$v$\frac{du}{dx}$dx
- where u and $\frac{dv}{dx}$ are chosen parts of the integrand.
- How to choose u and $\frac{dv}{dx}$

  I (Inverse trig)
  L (Logs)
  A (Algebra)
  T (Trig)
  E (e)
- u will be the operator which is higher in the ILATE rule. And the other will be taken as $\frac{dv}{dx}$
- To solve Integration by parts
- Choose u and $\frac{dv}{dx}$
- Then differentiate u to get $\frac{du}{dx}$ and integrate $\frac{dv}{dx}$ to get v
- Apply the integration by parts formula

Example

I = $\int$xe^{-2}dx

u = x

$\frac{du}{dx}$ = 1

$\frac{dv}{dx}$ = e^{-2}

$\int \frac{dv}{dx}$dx = $\int e^{-2}$dx

v = -$\frac{1}{2}e^{-2}$

$\int$u$\frac{dv}{dx}$dx = uv - $\int$v$\frac{du}{dx}$dx

$\int$xe^{-2}dx = x(-$\frac{1}{2}e^{-2}$) - $\int$-$\frac{1}{2}e^{-2}$dx

$\int$xe^{-2}dx = -$\frac{1}{2}$xe^{-2} +$\frac{1}{2}\int e^{-2}$dx

$\int$xe^{-2}dx = -$\frac{1}{2}$xe^{-2} +$\frac{1}{2}$(-$\frac{1}{2}e^{-2}$)dx

$\int$xe^{-2}dx = -$\frac{1}{2}$xe^{-2} - $\frac{1}{4}e^{-2x}$ + C

I = -$\frac{1}{2}$xe^{-2} - $\frac{1}{4}e^{-2x}$ + C

Numerical Solution of Equation

- ➔ There are 2 types of Functions; Divergent and Convergent. Convergent functions get more and more accurate as we increase the number of times its answer is substituted back, however, divergent function does not.

Proving that a root lies between 2 values (x^1 and x^2)

- ➔ Bring all the components of the function onto 1 side and let the other side be 0.
- ➔ Substitute one of the values in place of x and solve it to get an integer value.
- ➔ Then do the same with the other value and get another integer value.
- ➔ You will be able to observe that both the results will have different signs.
- ➔ Lastly, you can write a statement mentioning that, as the sign is changing, hence we can say that the root lies between x^1 and x^2.

Rearranging the equation

In these type of questions we can use 2 methods:

1. We can either identify which x in the equation has been made the subject and therefore take the original equation and separate that x by making it the subject.
2. We can take the 2nd equation and then convert it back to the original form of the equation.

Both these methods will prove that these 2 equations can be transformed into one another

Determining the exact root

This will mostly be the last part of the question and we will have mostly proved that the root lies between 2 numbers and that the equation is interchangeable to another form as per above.

- ➔ First in this part we will take the 2nd equation they provided us where x has been made the subject and then we will use a formula that is: Xn = ax(n+1) + b
- ➔ We can take any number between those 2 roots, most appropriate to take the mid value of those 2 roots.
- ➔ We will substitute that value of x into the equation and get an answer
- ➔ Make sure to note down these numbers with 2 extra dp as asked for the final answer. If the final answer is asked in 2dp the working should be in 4dp.
- ➔ We will insert back our first answer into the equation in place of x to get a 2nd answer
- ➔ We will keep on doing it until we get 3 consecutive same answers as per the dp in the final answer.

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Vectors

Vectors:

Direction Vector - XM - vector from X to M.

Position Vector - OM - vector relative to origin

FORMULA:- $XM = OM - OX$

Forms of Vector Equations

1) **Vector**
   $r = (\,position\ vector) + \lambda\,(direction\ vector)$

2) **Parametric**
   Open up the vector form.
   $x = 3 + 2\lambda$
   $y = 2 + 3\lambda$
   $z = 1 + 4\lambda$

$$\mathbf{r} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} + \lambda \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$$

3) **Cartesian**

$$\left(\frac{x-(point)}{(direction)}\right) = \frac{x-(3)}{2} = \frac{y-(2)}{3} = \frac{z-(1)}{4}$$

4) **Vector**
   $r = (\,position\ vector) + \lambda\,(direction\ vector)$

5) **Parametric**
   Open up the vector form.
   $x = 3 + 2\lambda$
   $y = 2 + 3\lambda$
   $z = 1 + 4\lambda$

6) **Cartesian**

$$\left(\frac{x-(point)}{(direction)}\right) = \frac{x-(3)}{2} = \frac{y-(2)}{3} = \frac{z-(1)}{4}$$

Magnitude

Use the direction vector in the formula to find out the magnitude/ distance of a line.

$AB = \sqrt{x^2 + y^2 + z^2}$

Dot product

$a \bullet b = |a||b|cos\theta$

$\begin{pmatrix} A \\ B \\ C \end{pmatrix} . \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}$ = (A x χ) + (B x Y) + (C x Z) = (magnitude of a) x (magnitude of b) x cos *(angle between 2 lines)

Two lines questions

1) Either both parallel
2) Or both perpendicular.

Questions based on parallel

- Have a factor in common

Questions based on perpendicular:

Find out if they intersect or not.

1) Equate parametric equations of both the lines and obtain 3 simultaneous equations.
2) Make λ the subject
3) Substitute into the other equation and find μ in terms of λ
4) Substitute both values in 3rd equation and see if it satisfies
5) If it satisfies, both intersect each other, or else are skew.
6) To find the point of intersection, input the values found in any equation.
7) For angle, use dot product formula.

Find out the shortest distance from point X to line Y.

1) Bring the equation to parametric form and use it as coordinates for an unknown point M.
2) Find direction vector of M and X .(XM = OM - OX)

3) Equate both direction vectors dot product to 0. (as perpendicular lines dot product is always 0)
4) Find the value of 'μ' and put it in the XM equation to obtain OX.
5) To find distance between X and M, use the distance formula.

Differential Equations

What are different differential equations?

It is basically an equation where you cannot make either x or y the subject. Thus, you have to integrate both with respect to X and continue to substitute and solve.

Solving Differential Equations step by step:

STEP 1:

-You are given data mentioning quantities directly proportional to each other, you make an equation with them (*define a constant, remove the proportional sign, substitute values given to find out constant*)

-Form an equation in the form $\frac{dy}{dx}$.

STEP 2:

-Multiply both the sides by dx. (cancel out on one)
-Bring all 'y' variables to the side with dy. Bring all 'x' variables to the side with dx.

STEP 3:

-Integrate both sides.
-You will get an equation. Add a constant *(+ c)*

STEP 4:

-Solve for the constant, substituting values given in question.
-Obtain the final equation
-Format your answer as asked by the examiner (*in terms of y or x*)

STEP 5: (advanced)

-You can be asked to find the value of y = ? for a certain value of x, substitute and answer.
-Or you can be asked the maximum or minimum value approached.

Complex Numbers

Definitions and Properties

- All the numbers that can be expressed as a certain distance in the real world i.e. can be found on the number line are known as real numbers. This includes 1, 107, -0.23 and π
- Imaginary numbers are numbers that include i.
  E.g. i, -0.33i and πi. Complex numbers include both a real and imaginary part. E.g. 2 + 3i
- "i" is defined as the square root of -1 i.e.
  $i = \sqrt{-1}$ or $i^2 = -1$
- Addition and subtraction follow as normal. The real parts are added separately and the imaginary parts are added separately.
- For multiplication and division, some properties of the imaginary number i is needed. The powers of i from 0 to 4 are:
  $i^0 = 1$
  $i^1 = i$
  $i^2 = -1$
  $i^3 = -i$
  $i^4 = 1$
  The pattern from i^0 to i^3 carries on. For example the fifth power is again i and sixth is -1. It also applies to the negative powers. For example $i^{-1} = -i$ and $i^{-2} = -1$.
- This pattern can also be useful in determining higher powers of i. Since $i^4 = 1$ and every 4 powers the pattern repeats, raising i to the power of 4 or 8 or 40 or 1000 equals 1. So $i^{1007} = i^{1004} \times i^3 = 1 \times i^3 = i^3 = -i$
- Another important tool for division is called 'realisation'. Using this a complex denominator can be changed to a real one. For a denominator in the form of (a + ib), multiply both the numerator and denominator by (a - ib). The denominator will now turn into $\{a^2 - (ib)^2\}$ which is then equal to (a^2+b^2).
  Here note the use of the formulas:
  $a^2 - b^2 = (a + b)(a - b)$ and $i^2 = -1$

Solved Examples

- Addition/ Subtraction:
  (3 + 2i) + (4.1 + 3i) = 7.1 + 5i
  (2 - i) - (3 +2i) = -1 -3i

  In general:
  (a + ib) + (c +id) = (a+c) + (b+d)i
  (a + ib) - (c + id) = (a-c) + (b-d)i
- Multiplication/ Division

  2(3 + 4i) = 6 + 8i
  (3 + 2i)(6 - i) = 3(6 - i) + 2i(6-i) = 18 - 3i + 12i + 2 = 20 + 9i

$$\frac{1+2i}{2} = 0.5 + i$$

$$\frac{3+5i}{1+2i} = \frac{3+5i}{1+2i} \times \frac{1-2i}{1-2i} = \frac{13-i}{5} = 2.6 + 0.2i$$

In general:

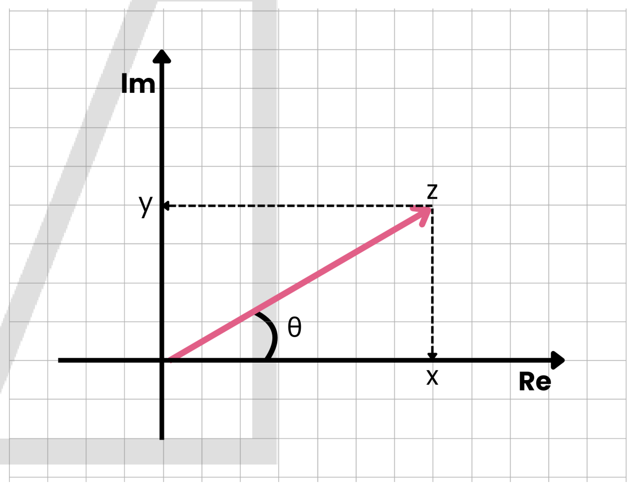
a(b + ci) = ab + aci
(a + bi)(c + di) = a(c + di) + bi(c + di)

$$\frac{a+bi}{c} = \frac{a}{c} + \frac{b}{c}i$$

$\frac{a+bi}{c+di} = \frac{a+bi}{c+di} \times \frac{c-di}{c-di} = \frac{(a+bi)(c-di)}{(c+di)(c-di)}$ here $(c + di)(c - di)$ becomes $c^2 + d^2$
and then we can use the above formula to solve.

The complex plane and Argand diagrams

General Form:
Z = x + iy
x: Real part
y: Imaginary part
Re: Real axis
Im: Imaginary axis



Modulus of Z
shown as r in the figure
It is the distance of Z from the origin, it can also be written as |Z|

$|Z| = \sqrt{x^2 + y^2}$

Argument of Z
It is shown with Q in the figure
Angle between Z and the positive real-axis, it can also be written as Arg z
Q = tan -1(y/x)

Conjugate of Z
Can be shown with line on top of Z or Z*
Z* = x - iy
(sign of imaginary part changes)

How to find Roots
Completing Square Method
if 1 root is -2+5i then other root will be its conjugate -2-5i

Finding square roots of complex numbers

Let $\sqrt{3 + 4i}$ = a + bi
Take square on both sides
$3 + 4i = a^2 - b^2 + 2abi$
$3 = a^2 - b^2$, $4 = 2ab$ ($a = 2/b$)
substitute to other equation
$3 = 4/b^2 - b^2$
$3b^2 = 4 - b^4$
$b^4 + 3b^2 - 4 = 0$
$b^2 = 1$, $b^2 = -4$
b = 1, b = 2i
a = 2, a = i
$\sqrt{3 + 4i}$ = 2 + i

Different forms of Complex Numbers

1. Cartesian Form
2. Polar Form
3. Exponential Form

1. Z = x + iy

2. Z = r(cosQ + isinQ)
r = |z| and Q = Arg z (can be in both degrees and Radians)

3. $Z = re^{i\theta}$
In this θ is in Radians

How to deal with Modulus and Argument while multiplying or dividing complex numbers?

Multiplication:
Arg (z1z2) = Arg (z_1) + Arg (z_2)
$|z_1z_2| = |z_1| \times |z_2|$

Division:
Arg (z_1/z_2) = Arg (z_1) - Arg (z_2)
$|z_1/z_2| = |z_1| \div |z_2|$

MOJZA

A Note from Mojza

These notes for Mathematics (9709) have been prepared by Team Mojza, covering the content for A Level 2022-24 syllabus. The content of these notes has been prepared with utmost care. We apologise for any issues overlooked; factual, grammatical or otherwise. We hope that you benefit from these and find them useful towards achieving your goals for your Cambridge examinations.

If you find any issues within these notes or have any feedback, please contact us at support@mojza.org.

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Authors:
Musbah Moid
Shahmeer Bukhari
Mehak Gurnani

Proofreaders:
Miraal Omer

Designers:
Fasiha Raza
Hasan Nawaz