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AS level

Paper 3 - Solutions

9709/03

BY TEAM MOJZA

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Modulus

Solve the equation

9709_s32_13 -q1

Solve the equation $|x-2| = \left|\frac{1}{3}x\right|$. [3]

$$(|x-2|)^2 = \left(\left|\frac{1}{3}x\right|\right)^2$$

$$x^2 - 4x + 4 = \frac{x^2}{9}$$

$$9x^2 - 36x + 36 = x^2$$

$$8x^2 - 36x + 36 = 0$$

$$2x^2 - 9x + 9 = 0$$

$$x = \frac{-(-9) \pm \sqrt{(-9)^2 - 4(2)(9)}}{2(2)}$$

$$x = \frac{9 \pm \sqrt{9}}{4}$$

$$x = \frac{9+3}{4}, \quad x = \frac{9-3}{4}$$

$$x = 3, \quad x = \frac{3}{2}$$

Solve the inequality

9709_s32_23 **-q1**

Solve the inequality $|5x - 3| < 2|3x - 7|$. [4]

$$(|5x-3|)^2 < (2|3x-7|)^2$$

$$25x^2 - 30x + 9 < 4(9x^2 - 42x + 49)$$

$$25x^2 - 30x + 9 < 36x^2 - 168x + 196$$

$$0 < 11x^2 - 138x + 187$$

$$x = \frac{-(-138) \pm \sqrt{(-138)^2 - 4(11)(187)}}{2(11)}$$

$$x = \frac{138 \pm \sqrt{10816}}{22}$$

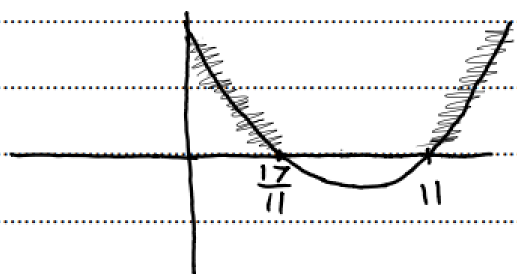
$$x = \frac{138 \pm 104}{22}$$

$$x = \frac{138+104}{22}, \quad x = \frac{138-104}{22}$$

$$x = 11, \quad x = \frac{17}{11}$$

crit values $= 11, \frac{17}{11}$

$x > 11, \quad x < \frac{17}{11}$



Partial fractions

Linear Form

9709_s31_10 -q8

Express $\dfrac{2}{(x+1)(x+3)}$ in partial fractions. [2]

$$\frac{2}{(x+1)(x+3)} \equiv \frac{A}{(x+1)} + \frac{B}{(x+3)}$$

$$\frac{2}{(x+1)(x+3)} \equiv \frac{A(x+3) + B(x+1)}{(x+1)(x+3)}$$

$$2 \equiv A(x+3) + B(x+1)$$

$x+1 = 0$, $x = -1$:

$$2 = A(-1+3) + B(-1+1)$$
$$2 = 2A$$
$$A = 1$$

$x+3 = 0$, $x = -3$:

$$2 = A(-3+3) + B(-3+1)$$
$$2 = -2B$$
$$B = -1$$

$$\frac{2}{(x+1)(x+3)} = \frac{1}{(x+1)} - \frac{1}{(x+3)}$$

Repeated Form

9709_s33_15 -q10

Let $f(x) = \dfrac{11x+7}{(2x-1)(x+2)^2}$.

(i) Express $f(x)$ in partial fractions. [5]

$$\frac{11x+7}{(2x-1)(x+2)^2} \equiv \frac{A}{(2x-1)} + \frac{B}{(x+2)} + \frac{C}{(x+2)^2}$$

$$\frac{11x+7}{(2x-1)(x+2)^2} \equiv \frac{A(x+2)^3 + B(2x-1)(x+2)^2 + C(2x-1)(x+2)}{(2x-1)(x+2)^3}$$

$$11x+7 \equiv \frac{A(x+2)^3 + B(2x-1)(x+2)^2 + C(2x-1)(x+2)}{(x+2)}$$

$$11x+7 \equiv A(x+2)^2 + B(2x-1)(x+2) + C(2x-1)$$

$x+2=0$, $x=-2$:

$11(-2)+7 = C(2(-2)-1)$

$-15 = -5C$

$C = 3$

$2x-1=0$, $2x=1$, $x=\frac{1}{2}$:

$11(\frac{1}{2})+7 = A(\frac{1}{2}+2)^2$

$\frac{25}{2} = \frac{25}{4}A$

$A = 2$

$x=0$:

$7 = A(2)^2 + B(-1)(2) + C(-1)$

$7 = (2)(4) - 2B - 3$

$2B = -2$

$B = -1$

$$\frac{2}{2x-1} - \frac{1}{x+2} + \frac{3}{(x+2)^2}$$

Quadratic Form

9709_s32_15 -q8

Let $f(x) = \dfrac{5x^2 + x + 6}{(3 - 2x)(x^2 + 4)}$.

(i) Express $f(x)$ in partial fractions. [5]

$$\frac{5x^2 + x + 6}{(3-2x)(x^2+4)} \equiv \frac{A}{(3-2x)} + \frac{Bx + C}{(x^2+4)}$$

$$\frac{5x^2 + x + 6}{(3-2x)(x^2+4)} \equiv \frac{A(x^2+4) + (Bx+C)(3-2x)}{(3-2x)(x^2+4)}$$

$$5x^2 + x + 6 \equiv A(x^2+4) + (Bx+C)(3-2x)$$

$3 - 2x = 0$

$3 = 2x$

$x = \frac{3}{2}$

$$5\left(\tfrac{3}{2}\right)^2 + \tfrac{3}{2} + 6 = A\left(\left(\tfrac{3}{2}\right)^2 + 4\right)$$

$$\frac{45}{4} + \frac{3}{2} + 6 = \frac{25}{4}A$$

$$\frac{75}{4} = \frac{25}{4}A$$

$$A = 3$$

$x = 0$

$$5(0)^2 + 0 + 6 = A(0^2 + 4) + (B(0) + C)(3 - 2(0))$$

$$6 = 4(3) + 3C$$

$$-6 = 3C$$

$$-2 = C$$

$x = 1$

$$5(1)^2 + 1 + 6 = A(1^2 + 4) + (B(1) + C)(3 - 2(1))$$

$$12 = 5A + B + C$$

$$12 = 5(3) + B + (-2)$$

$$12 - 15 + 2 = B$$

$$B = -1$$

$$\frac{3}{(3-2x)} - \frac{x+2}{(x^2+4)}$$

Improper fraction

9709_s03_08 -q7

Let $f(x) \equiv \dfrac{x^2 + 3x + 3}{(x+1)(x+3)}$.

(i) Express $f(x)$ in partial fractions. [5]

$(x+1)(x+3) = x^2 + 3x + x + 3$

$= x^2 + 4x + 3$

$$\begin{array}{r|l} & 1 \\ \hline x^2+4x+3 & x^2 + 3x + 3 \\ & -(x^2 + 4x + 3) \\ \hline & -x \end{array}$$

$1 - \dfrac{x}{(x+1)(x+3)}$

$\dfrac{x}{(x+1)(x+3)} \equiv \dfrac{A}{(x+1)} + \dfrac{B}{(x+3)}$

$\dfrac{x}{(x+1)(x+3)} \equiv \dfrac{A(x+3) + B(x+1)}{(x+1)(x+3)}$

$x \equiv A(x+3) + B(x+1)$

$x + 1 = 0$, $x = -1$ | $-1 = A(-1+3)$

$-1 = 2A$

$A = -\frac{1}{2}$

$x + 3 = 0$, $x = -3$ | $-3 = B(-3+1)$

$-3 = -2B$

$B = \frac{3}{2}$

$1 - \left[\dfrac{-\frac{1}{2}}{(x+1)} + \dfrac{\frac{3}{2}}{(x+3)}\right]$

$1 + \dfrac{\frac{1}{2}}{(x+1)} - \dfrac{\frac{3}{2}}{(x+3)}$

Binomial theorem

Expand up till X-term

9709_s03_05 -q1

Expand $(1+4x)^{-\frac{1}{2}}$ in ascending powers of x, up to and including the term in x^3, simplifying the coefficients. [4]

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots$$

$$(1+4x)^{-\frac{1}{2}} = 1 + \left(-\tfrac{1}{2}\right)(4x) + \frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)}{2!}(4x)^2 + \frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)\left(-\frac{1}{2}-2\right)}{3!}(4x)^3$$

$$(1+4x)^{-\frac{1}{2}} = 1 - 2x + \frac{3}{8}(16x^2) - \frac{5}{16}(64x^3)$$

$$(1+4x)^{-\frac{1}{2}} = 1 - 2x + 6x^2 - 20x^3$$

9709_s03_07 -q1

Expand $(2+3x)^{-2}$ in ascending powers of x, up to and including the term in x^2, simplifying the coefficients. [4]

$$(2+3x)^{-2}$$

$$\left(2\left(1+\tfrac{3}{2}x\right)\right)^{-2}$$

$$\frac{1}{4}\left(1+\tfrac{3}{2}x\right)^{-2}$$

$$\left(1+\tfrac{3}{2}x\right)^{-2} = 1 - 2\left(\frac{3}{2}x\right) + \frac{(-2)(-2-1)}{2!}$$

$$= 1 - 3x + \frac{27}{4}x^2$$

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$$\frac{1}{4}\left(1+\frac{3}{2}x\right)^{-2}$$

$$\frac{1}{4}\left(1 - 3x + \frac{27}{4}x^2\right)$$

$$\frac{1}{4} - \frac{3}{4}x + \frac{27}{16}x^2$$

Logarithmic and Exponential Functions

Equation solving

9709_s03_04 -q4

(i) Show that if $y = 2^x$, then the equation

$$2^x - 2^{-x} = 1$$

can be written as a quadratic equation in y. [2]

$y - y^{-1} = 1$

$y - \frac{1}{y} = 1$

$y^2 - 1 = y$

$y^2 - y - 1 = 0$

(ii) Hence solve the equation

$$2^x - 2^{-x} = 1.$$ [4]

$$y^2 - y - 1 = 0$$

$$y = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(-1)}}{2(1)}$$

$$y = \frac{1 \pm \sqrt{5}}{2}$$

$$y = \frac{1 + \sqrt{5}}{2} \quad , \quad y = \frac{1 - \sqrt{5}}{2}$$

$$2^x = \frac{1 + \sqrt{5}}{2} \quad , \quad 2^x = \frac{1 - \sqrt{5}}{2}$$

$$\ln 2^x = \ln\left(\frac{1 + \sqrt{5}}{2}\right) \quad , \quad \ln 2^x = \ln\left(\frac{1 - \sqrt{5}}{2}\right)$$

$$x \ln 2 = \ln\left(\frac{1 + \sqrt{5}}{2}\right) \qquad \text{no solution}$$

$$x = \ln\left(\frac{1 + \sqrt{5}}{2}\right) \div \ln 2$$

$$x = 0.694$$

Expressing y in terms of x

9709_s06_p3 -q1

Given that $x = 4(3^{-y})$, express y in terms of x. [3]

$$x = 4(3^{-y})$$

$$\frac{x}{4} = 3^{-y}$$

$$\ln\left(\frac{x}{4}\right) = \ln 3^{-y}$$

$$\ln\left(\frac{x}{4}\right) = -y \ln 3$$

$$\ln\left(\frac{x}{4}\right) \div \ln 3 = -y$$

$$y = -\frac{\ln\left(\frac{x}{4}\right)}{\ln(3)}$$

Trigonometry

Rcos($\theta \pm \alpha$) form

9709_s06_p3 -q4

(i) Express $7\cos\theta + 24\sin\theta$ in the form $R\cos(\theta - \alpha)$, where $R > 0$ and $0° < \alpha < 90°$, giving the exact value of R and the value of α correct to 2 decimal places. [3]

$$7\cos\theta + 24\sin\theta \equiv R\cos(\theta - \alpha)$$

$$7\cos\theta + 24\sin\theta \equiv R[\cos\theta\cos\alpha + \sin\theta\sin\alpha]$$

$$7\cos\theta + 24\sin\theta \equiv R\cos\theta\cos\alpha + R\sin\theta\sin\alpha$$

$$7\cos\theta = R\cos\theta\cos\alpha \quad , \quad 24\sin\theta = R\sin\theta\sin\alpha$$

$$7 = R\cos\alpha \quad , \quad 24 = R\sin\alpha$$

$$\frac{R\sin\alpha}{R\cos\alpha} = \frac{24}{7}$$

$$\frac{\sin\alpha}{\cos\alpha} = \frac{24}{7}$$

$$\tan\alpha = \frac{24}{7}$$

$$\alpha = \tan^{-1}\left(\frac{24}{7}\right)$$

$$\alpha = 73.74°$$

$$R^2\cos^2\alpha = 7^2$$

$$R^2\sin^2\alpha = 24^2$$

$$R^2\cos^2\alpha + R^2\sin^2\alpha = 7^2 + 24^2$$

$$R^2(\sin^2\alpha + \cos^2\alpha) = 625$$

$$R^2(1) = 625$$

$$R = 25$$

$$7\cos\theta + 24\sin\theta \equiv 25\cos(\theta - 73.74)$$

Rsin(θ ± α) form

9709_w32_19 -q4

(i) Express $(\sqrt{6})\sin x + \cos x$ in the form $R\sin(x + \alpha)$, where $R > 0$ and $0° < \alpha < 90°$. State the exact value of R and give α correct to 3 decimal places. [3]

$$(\sqrt{6})\sin x + \cos x \equiv R\sin(x+\alpha)$$

$$(\sqrt{6})\sin x + \cos x \equiv R[\sin x\cos\alpha + \cos x\sin\alpha]$$

$$(\sqrt{6})\sin x + \cos x \equiv R\sin x\cos\alpha + R\cos x\sin\alpha$$

$$(\sqrt{6})\sin x = R\sin x\cos\alpha, \quad \cos x = R\cos x\sin\alpha$$

$$\sqrt{6} = R\cos\alpha, \quad 1 = R\sin\alpha$$

$$\frac{R\sin\alpha}{R\cos\alpha} = \frac{1}{\sqrt{6}}$$

$$\frac{\sin\alpha}{\cos\alpha} = \frac{1}{\sqrt{6}}$$

$$\tan\alpha = \frac{1}{\sqrt{6}}$$

$$\alpha = \tan^{-1}\left(\frac{1}{\sqrt{6}}\right)$$

$$\alpha = 22.208°$$

$$R^2\cos^2\alpha = (\sqrt{6})^2$$

$$R^2\sin^2\alpha = 1^2$$

$$R^2\cos^2\alpha + R^2\sin^2\alpha = (\sqrt{6})^2 + 1^2$$

$$R^2(\sin^2\alpha + \cos^2\alpha) = 7$$

$$R^2(1) = 7$$

$$R = \sqrt{7}$$

$$(\sqrt{6})\sin x + \cos x \equiv \sqrt{7}\cos(\theta - 22.208)$$

Compound angle form

9709_w03_04 -q4

(i) Show that the equation

$$\tan(45^\circ + x) = 2\tan(45^\circ - x)$$

can be written in the form

$$\tan^2 x - 6\tan x + 1 = 0.$$ [4]

$$\tan(45 + x) = 2\tan(45 - x)$$

$$\frac{\tan 45 + \tan x}{1 - \tan 45 \tan x} = 2\left[\frac{\tan 45 - \tan x}{1 + \tan 45 \tan x}\right]$$

$$\frac{1 + \tan x}{1 - \tan x} = 2\left[\frac{1 - \tan x}{1 + \tan x}\right]$$

$$(1 + \tan x)^2 = 2(1 - \tan x)^2$$

$$1 + 2\tan x + \tan^2 x = 2(1 - 2\tan x + \tan^2 x)$$

$$1 + 2\tan x + \tan^2 x = 2 - 4\tan x + 2\tan^2 x$$

$$0 = \tan^2 x - 6\tan x + 1$$

Compound angle expanding

9709_s33_p12 -q6

It is given that $\tan 3x = k\tan x$, where k is a constant and $\tan x \neq 0$.

(i) By first expanding $\tan(2x + x)$, show that

$$(3k - 1)\tan^2 x = k - 3.$$ [4]

$$\tan 3x = k\tan x$$

$$\tan(2x + x) = k\tan x$$

$$\frac{\tan 2x + \tan x}{1 - \tan 2x \tan x} = k \tan x$$

$$\tan 2x + \tan x = k\tan x\,(1 - \tan 2x \tan x)$$

$$\frac{2\tan x}{1-\tan^2 x} + \tan x = k\tan x\left(1 - \frac{2\tan x}{1-\tan^2 x} \cdot \tan x\right)$$

$$\cancel{\tan x}\left(\frac{2}{1-\tan^2 x} + 1\right) = k\cancel{\tan x}\left(1 - \frac{2\tan^2 x}{1-\tan^2 x}\right)$$

$$\frac{2 + 1 - \tan^2 x}{1-\tan^2 x} = k\left(\frac{1 - \tan^2 x - 2\tan^2 x}{1-\tan^2 x}\right)$$

$$3 - \tan^2 x = k(1 - 3\tan^2 x)$$

$$3 - \tan^2 x = k - 3k\tan^2 x$$

$$3k\tan^2 x - \tan^2 x = k - 3$$

$$(3k-1)(\tan^2 x) = k - 3$$

Prove the identity

9709_s03_05 -q6

(i) Prove the identity

$$\cos 4\theta + 4\cos 2\theta \equiv 8\cos^4\theta - 3. \qquad [4]$$

$$\cos 4\theta + 4\cos 2\theta$$

$$2\cos^2 2\theta - 1 + 4(2\cos^2\theta - 1)$$

$$2\cos^2 2\theta - 1 + 8\cos^2\theta - 4$$

$$2(\cos 2\theta)^2 - 1 + 8\cos^2\theta - 4$$

$$2(2\cos^2\theta - 1)^2 - 1 + 8\cos^2\theta - 4$$

$$2(4\cos^4\theta - 4\cos^2\theta + 1) - 1 + 8\cos^2\theta - 4$$

$$8\cos^4\theta - 8\cos^2\theta + 2 - 1 + 8\cos^2\theta - 4$$

$$8\cos^4\theta - 3$$

9709_s03_09 -q3

(i) Prove the identity $\operatorname{cosec} 2\theta + \cot 2\theta \equiv \cot \theta$. [3]

$$\frac{1}{\sin 2\theta} + \frac{\cos 2\theta}{\sin 2\theta}$$

$$\frac{1 + \cos 2\theta}{\sin 2\theta}$$

$$\frac{1 + 2\cos^2\theta - 1}{2\sin\theta\cos\theta}$$

$$\frac{2\cos^2\theta}{2\sin\theta\cos\theta}$$

$$\frac{\cos\theta}{\sin\theta}$$

$$\cot\theta$$

Differentiation

dy/dx and Stationary Points

9709_w03_06 -q3

The curve with equation $y = 6e^x - e^{3x}$ has one stationary point.

(i) Find the x-coordinate of this point. [4]

$$y = 6e^x - e^{3x}$$

$$\frac{dy}{dx} = 6e^x(1) - e^{3x}(3)$$

$$\frac{dy}{dx} = 6e^x - 3e^{3x}$$

$$0 = 6e^x - 3e^{3x}$$

$$3e^{3x} = 6e^x$$

$$\frac{e^{3x}}{e^x} = \frac{6}{3}$$

$$e^{2x} = 2$$

$$\ln e^{2x} = \ln 2$$

$$2x(1) = \ln 2$$

$$x = \frac{\ln 2}{2}$$

$$x = 0.347$$

(ii) Determine whether this point is a maximum or a minimum point. [2]

$$\frac{dy}{dx} = 6e^{x} - 3e^{3x}$$

$$\frac{d^2y}{dx^2} = 6e^{x}(1) - 3e^{3x}(3)$$

$$\frac{d^2y}{dx^2} = 6e^{x} - 9e^{3x}$$

$$x = 0.347$$

$$\frac{d^2y}{dx^2} = 6e^{0.347} - 9e^{3(0.347)}$$

$$\frac{d^2y}{dx^2} = -16.99 \quad (\text{Maximum point})$$

9709_s33_11 -q2

The curve $y = \dfrac{\ln x}{x^3}$ has one stationary point. Find the x-coordinate of this point. [4]

$$y = \frac{\ln x}{x^3}$$

$$\frac{dy}{dx} = \frac{x^3\left(\frac{1}{x}\right) - \ln x\left(3x^2\right)}{(x^3)^2}$$

$$\frac{dy}{dx} = \frac{x^2 - 3x^2 \ln x}{x^6}$$

$$\frac{dy}{dx} = \frac{x^2(1 - 3\ln x)}{x^6}$$

$$\frac{dy}{dx} = \frac{1 - 3\ln x}{x^4}$$

$$0 = \frac{1 - 3\ln x}{x^4}$$

$$0 = 1 - 3\ln x$$

$$3\ln x = 1$$

$$\ln x = \frac{1}{3}$$

$$e^{\ln x} = e^{\frac{1}{3}}$$

$$x = e^{\frac{1}{3}}$$

Parametric equations

9709_s33_12 -q3

The parametric equations of a curve are

$$x = \sin 2\theta - \theta, \qquad y = \cos 2\theta + 2\sin\theta.$$

Show that $\dfrac{dy}{dx} = \dfrac{2\cos\theta}{1 + 2\sin\theta}$. [5]

$x = \sin 2\theta - \theta$ $\qquad$ $y = \cos 2\theta + 2\sin\theta$

$\frac{dx}{d\theta} = (\cos 2\theta)(2) - 1$ $\qquad$ $\frac{dy}{d\theta} = (-\sin 2\theta)(2) + 2\cos\theta(1)$

$\frac{dx}{d\theta} = 2\cos 2\theta - 1$ $\qquad$ $\frac{dy}{d\theta} = 2\cos\theta - 2\sin 2\theta$

$$\frac{dy}{dx} = \frac{2\cos\theta - 2\sin 2\theta}{2\cos 2\theta - 1}$$

$$\frac{dy}{dx} = \frac{2\cos\theta - 2(2\sin\theta\cos\theta)}{2(1-2\sin^2\theta) - 1}$$

$$\frac{dy}{dx} = \frac{2\cos\theta(1-2\sin\theta)}{2(1-2\sin^2\theta) - 1}$$

$$\frac{dy}{dx} = \frac{2\cos\theta(1-2\sin\theta)}{2-4\sin^2\theta - 1}$$

$$\frac{dy}{dx} = \frac{2\cos\theta(1-2\sin\theta)}{1-4\sin^2\theta}$$

$$\frac{dy}{dx} = \frac{2\cos\theta(1-2\sin\theta)}{(1)^2 - (2\sin\theta)^2}$$

$$\frac{dy}{dx} = \frac{2\cos\theta\cancel{(1-2\sin\theta)}}{(1+2\sin\theta)\cancel{(1-2\sin\theta)}}$$

$$\frac{dy}{dx} = \frac{2\cos\theta}{1+2\sin\theta}$$

Implicit differentiation

9709_w03_06 -q6

The equation of a curve is $x^3 + 2y^3 = 3xy$.

(i) Show that $\frac{dy}{dx} = \frac{y - x^2}{2y^2 - x}$. [4]

$$x^3 + 2y^3 = 3xy$$

$$3x^2(1) + 2(3)y^2\left(\frac{dy}{dx}\right) = 3\left[x\left(\frac{dy}{dx}\right) + y(1)\right]$$

$$3x^2 + 6y^2\frac{dy}{dx} = 3x\frac{dy}{dx} + 3y$$

$$6y^2\frac{dy}{dx} - 3x\frac{dy}{dx} = 3y - 3x^2$$

$$\frac{dy}{dx}\left(6y^2 - 3x\right) = 3y - 3x^2$$

$$\frac{dy}{dx} = \frac{3y - 3x^2}{6y^2 - 3x}$$

$$\frac{dy}{dx} = \frac{y - x^2}{2y^2 - x}$$

(ii) Find the coordinates of the point, other than the origin, where the curve has a tangent which is parallel to the x-axis. [5]

$$\frac{dy}{dx} = 0$$

$$\frac{y - x^2}{2y^2 - x} = 0$$

$$y - x^2 = 0$$

$$y = x^2$$

$$x^3 + 2y^3 = 3xy$$

$$x^3 + 2(x^2)^3 = 3x(x^2)$$

$$x^3 + 2x^6 = 3x^3$$

$$2x^6 = 2x^3$$

$$\frac{x^6}{x^3} = \frac{2}{2}$$

$$x^3 = 1$$

$$x = 1$$

$$y = x^2$$

$$y = (1)^2$$

$$y = 1$$

$$(1, 1)$$

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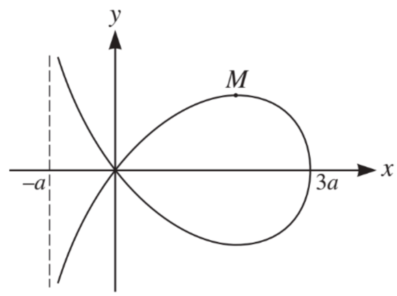
9709_s32_13 -q5



The diagram shows the curve with equation

$$x^3 + xy^2 + ay^2 - 3ax^2 = 0,$$

where a is a positive constant. The maximum point on the curve is M. Find the x-coordinate of M in terms of a. [6]

$$x^3 + xy^2 + ay^2 - 3ax^2 = 0$$

$$3x^2(1) + \left[x\left(2y\frac{dy}{dx}\right) + y^2(1)\right] + 2ay\left(\frac{dy}{dx}\right) - 3ax(2)$$

$$3x^2 + 2xy\frac{dy}{dx} + y^2 + 2ay\frac{dy}{dx} - 6ax$$

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$$\frac{dy}{dx} = 0$$

$$3x^2 + y^2 - 6ax = 0$$

$$y^2 = 6ax - 3x^2$$

$$x^3 + x(6ax - 3x^2) + a(6ax - 3x^2) - 3ax^2 = 0$$

$$x^3 + 6ax^2 - 3x^3 + 6a^2x - 3ax^2 - 3ax^2 = 0$$

$$6a^2x - 2x^3 = 0$$

$$2x^3 = 6a^2x$$

$$2x^2 = 6a^2$$

$$x^2 = 3a^2$$

$$x = \sqrt{3a^2}$$

$$x = \sqrt{3}\,a$$

Numerical Solution of Equations

Prove α lies between x and y

9709_s32_23 -q6

The equation $\cot \frac{1}{2}x = 3x$ has one root in the interval $0 < x < \pi$, denoted by α.

(a) Show by calculation that α lies between 0.5 and 1. [2]

$\cot \frac{1}{2}x - 3x = 0$

$y = \cot \frac{1}{2}x - 3x$

$x = 0.5$,

$y = \cot \frac{1}{2}(0.5) - 3(0.5)$

$y = 2.42$

$x = 1$,

$y = \cot \frac{1}{2}(1) - 3(1)$

$y = -1.17$

Sign changes between 0.5 and 1

Show that the iterative formula converges to α

9709_s32_23 -q6

(b) Show that, if a sequence of positive values given by the iterative formula

$$x_{n+1} = \frac{1}{3}\left(x_n + 4\tan^{-1}\left(\frac{1}{3x_n}\right)\right)$$

converges, then it converges to α. [2]

$$x = \frac{1}{3}\left(x + 4\tan^{-1}\left(\frac{1}{3x}\right)\right)$$

$$3x = x + 4\tan^{-1}\left(\frac{1}{3x}\right)$$

$$2x = 4\tan^{-1}\left(\frac{1}{3x}\right)$$

$$\frac{1}{2}x = \tan^{-1}\left(\frac{1}{3x}\right)$$

$$\tan\frac{1}{2}x = \frac{1}{3x}$$

$$3x = \frac{1}{\tan\frac{1}{2}x}$$

$$3x = \cot\frac{1}{2}x$$

Calculate α correct to x d.p

9709_s32_23 -q6

(c) Use this iterative formula to calculate α correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

$$x_1 = \frac{(0.5+1)}{2} = 0.75$$

$$x_2 = \frac{1}{3}\left(0.75 + 4\tan^{-1}\left(\frac{1}{3(0.75)}\right)\right) = 0.8076$$

$$x_3 = 0.7911$$

$$x_4 = 0.7954$$

$$x_5 = 0.7943$$

$$x_6 = 0.7946$$

$$x_7 = 0.7945$$

$$x_8 = 0.7945$$

$$x = 0.79$$

Prove α has only one root (graphs)

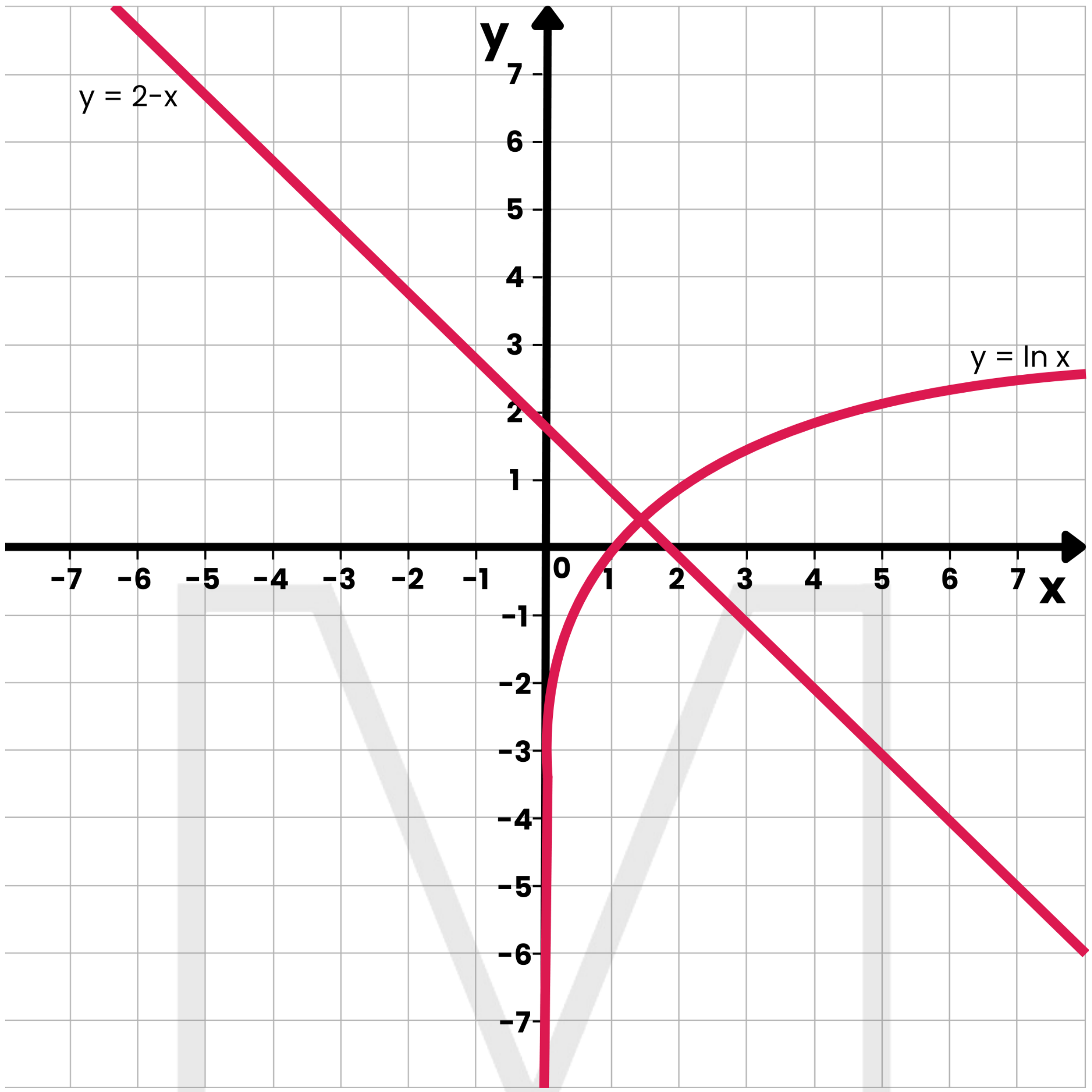
9709_w03_07 -q6

(i) By sketching a suitable pair of graphs, show that the equation

$$2 - x = \ln x$$

has only one root. [2]

The graphs $y = \ln x$ and $y = 2 - x$ only intersect at one point.


y
x
0
y = 2-x
y = ln x
-7
-6
-5
-4
-3
-2
-1
1
2
3
4
5
6
7

Vectors

Show that the lines do not intersect

9709_w03_04 -q9

The lines l and m have vector equations

$$\mathbf{r} = 2\mathbf{i} - \mathbf{j} + 4\mathbf{k} + s(\mathbf{i} + \mathbf{j} - \mathbf{k}) \qquad \text{and} \qquad \mathbf{r} = -2\mathbf{i} + 2\mathbf{j} + \mathbf{k} + t(-2\mathbf{i} + \mathbf{j} + \mathbf{k})$$

respectively.

(i) Show that l and m do not intersect. [4]

$$r = \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix} + s\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} , \qquad r = \begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix} + t\begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$$

$x = 2 + s$, $x = -2 - 2t$

$y = -1 + s$, $y = 2 + t$

$z = 4 - s$, $z = 1 + t$

$2 + s = -2 - 2t$ | $-1 + s = 2 + t$

$s = -4 - 2t$ | $s = 3 + t$

$-4 - 2t = 3 + t$

$-7 = 3t$

$t = -\frac{7}{3}$

$s = 3 + \left(-\frac{7}{3}\right)$

$s = \frac{2}{3}$

$4 - s = 1 + t$

$3 = s + t$

$3 = \frac{2}{3} + \left(-\frac{7}{3}\right)$

$3 \neq -\frac{5}{3}$

Find vector equation for line

9709_s03_06 -q

The points A and B have position vectors, relative to the origin O, given by

$$\overrightarrow{OA} = \begin{pmatrix} -1 \\ 3 \\ 5 \end{pmatrix} \quad \text{and} \quad \overrightarrow{OB} = \begin{pmatrix} 3 \\ -1 \\ -4 \end{pmatrix}.$$

The line l passes through A and is parallel to OB. The point N is the foot of the perpendicular from B to l.

(i) State a vector equation for the line l. [1]

$$r = a + \lambda m$$

$$m = \overrightarrow{OB} = \begin{pmatrix} 3 \\ -1 \\ -4 \end{pmatrix}, \quad a = \overrightarrow{OA} = \begin{pmatrix} -1 \\ 3 \\ 5 \end{pmatrix}$$

$$l = \begin{pmatrix} -1 \\ 3 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -1 \\ -4 \end{pmatrix}$$

Find the position vector

9709_s23_p32 -q11

(ii) Find the position vector of N and show that $BN = 3$. [6]

$$l = \begin{pmatrix} -1+3\lambda \\ 3-\lambda \\ 5-4\lambda \end{pmatrix}$$

$$\overrightarrow{BN} = \overrightarrow{ON} - \overrightarrow{OB}$$

$$\overrightarrow{BN} = \begin{pmatrix} -1+3\lambda \\ 3-\lambda \\ 5-4\lambda \end{pmatrix} - \begin{pmatrix} 3 \\ -1 \\ -4 \end{pmatrix}$$

$$\overrightarrow{BN} = \begin{pmatrix} -1+3\lambda-3 \\ 3-\lambda+1 \\ 5-4\lambda+4 \end{pmatrix}$$

$$\overrightarrow{BN} = \begin{pmatrix} -4+3\lambda \\ 4-\lambda \\ 9-4\lambda \end{pmatrix}$$

$$\overrightarrow{BN} \cdot \vec{m} = 0$$

$$\begin{pmatrix} -4+3\lambda \\ 4-\lambda \\ 9-4\lambda \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -1 \\ -4 \end{pmatrix} = 0$$

$$3(-4+3\lambda) - 1(4-\lambda) - 4(9-4\lambda) = 0$$

$$-12 + 9\lambda - 4 + \lambda - 36 + 16\lambda = 0$$

$$52 = 26\lambda$$

$$\lambda = 2$$

$$\overrightarrow{BN} = \begin{pmatrix} -4+3(2) \\ 4-(2) \\ 9-4(2) \end{pmatrix}$$

$$\overrightarrow{BN} = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$$

$$|\overrightarrow{BN}| = \sqrt{(2)^2+(2)^2+(1)^2} = \sqrt{9} = 3$$

MOJZA

Show that the lines are skew

9709_m21_p32 -q7

Two lines have equations $\mathbf{r} = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} + s\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix} + t\begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix}$.

(a) Show that the lines are skew. [5]

$x = 1 + 2s$, $x = 2 + t$

$y = 3 - s$, $y = 1 - t$

$z = 2 + 3s$, $z = 4 + 4t$

$1 + 2s = 2 + t$ | $3 - s = 1 - t$

$t = 2s - 1$ | $t = s - 2$

$2s - 1 = s - 2$

$s = -1$

$t = 2(-1) - 1$

$t = -3$

$2 + 3s = 4 + 4t$

$2 + 3(-1) = 4 + 4(-3)$

$-1 = -8$ skew

MOJZA

Calculate angle

9709_m21_p32 -q7

(b) Find the acute angle between the directions of the two lines. [3]

$$m_1 = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} \qquad m_2 = \begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix}$$

$$m_1 \cdot m_2 = |m_1||m_2| \cos\theta$$

$$\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix} = \left(\sqrt{(2)^2+(-1)^2+(3)^2}\right)\left(\sqrt{(1)^2+(-1)^2+4^2}\right)\cos\theta$$

$$2(1) - 1(-1) + 3(4) = (\sqrt{14})(\sqrt{18})\cos\theta$$

$$15 = \sqrt{14}\sqrt{18}\cos\theta$$

$$\frac{15}{\sqrt{14}\sqrt{18}} = \cos\theta$$

$$\theta = \cos^{-1}\left(\frac{15}{\sqrt{14}\sqrt{18}}\right)$$

$$\theta = 19.1^\circ$$

Differential equations

Expression for y in terms of x

9709_w11_p33 -q7

During an experiment, the number of organisms present at time t days is denoted by N, where N is treated as a continuous variable. It is given that

$$\frac{dN}{dt} = 1.2e^{-0.02t}N^{0.5}.$$

When $t = 0$, the number of organisms present is 100.

(i) Find an expression for N in terms of t. [6]

$$\frac{dN}{dt} = 1.2e^{-0.02t}N^{0.5}$$

$$\int \frac{1}{N^{0.5}}\,dN = 1.2\int e^{-0.02t}\,dt$$

$$\int \frac{N^{-0.5+1}}{-0.5+1}\,dN = 1.2\int \frac{e^{-0.02t}}{-0.02}\,dt$$

$$2N^{0.5} = 1.2(-50e^{-0.02t}) + C$$

$$2N^{0.5} = -60e^{-0.02t} + C$$

$$t = 0, \quad N = 100$$

$$2(100)^{0.5} = -60e^{-0.02(0)} + C$$

$$20 + 60 = C$$

$$C = 80$$

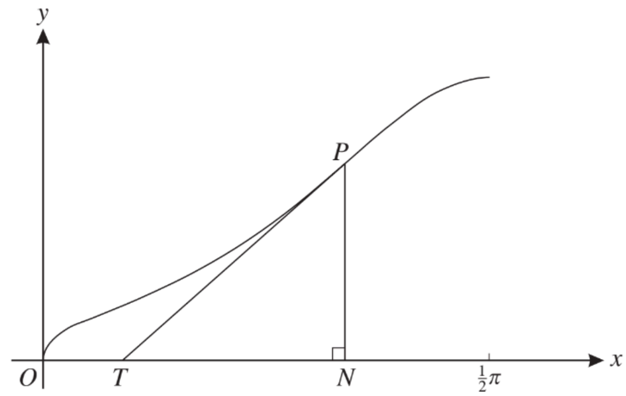
$$2N^{0.5} = -60e^{-0.02t} + 80$$

$$N^{0.5} = -30e^{-0.02t} + 40$$

$$N = (-30e^{-0.02t} + 40)^2$$

Solve the differential equation

9709_s03_08 -q



In the diagram the tangent to a curve at a general point P with coordinates (x, y) meets the x-axis at T. The point N on the x-axis is such that PN is perpendicular to the x-axis. The curve is such that, for all values of x in the interval $0 < x < \frac{1}{2}\pi$, the area of triangle PTN is equal to $\tan x$, where x is in radians.

(i) Using the fact that the gradient of the curve at P is $\frac{PN}{TN}$, show that

$$\frac{dy}{dx} = \frac{1}{2}y^2 \cot x. \qquad [3]$$

$\frac{dy}{dx} = \frac{PN}{TN}$

$\frac{dy}{dx} = \frac{y}{\frac{2\tan x}{y}}$

$\frac{dy}{dx} = \frac{y^2}{2\tan x}$

Area of PTN $= \frac{1}{2}(TN)(PN)$

$\tan x = \frac{1}{2}(TN)(y)$

$\frac{2\tan x}{y} = TN$

$$\frac{dy}{dx} = \frac{1}{2}y^2\cot x$$

(ii) Given that $y = 2$ when $x = \frac{1}{6}\pi$, solve this differential equation to find the equation of the curve, expressing y in terms of x. [6]

$$\frac{dy}{dx} = \frac{1}{2} y^2 \cot x$$

$$\int \frac{1}{y^2}\, dy = \frac{1}{2} \int \cot x \, dx$$

$$\int y^{-2}\, dy = \frac{1}{2} \int \frac{\cos x}{\sin x}\, dx$$

$$\frac{y^{-1}}{-1} = \frac{1}{2} \ln(\sin x) + C$$

$$-\frac{1}{y} = \frac{1}{2} \ln(\sin x) + C$$

$$y = 2 \quad , \quad x = \frac{\pi}{6}$$

$$-\frac{1}{2} = \frac{1}{2} \ln\left(\sin\left(\frac{\pi}{6}\right)\right) + C$$

$$\frac{-1}{2} = \frac{1}{2} \ln(2^{-1}) + C$$

$$-\frac{1}{2} = -\frac{1}{2} \ln 2 + C$$

$$C = \frac{1}{2} \ln 2 - \frac{1}{2}$$

$$-\frac{1}{y} = \frac{1}{2} \ln(\sin x) + \frac{1}{2} \ln 2 - \frac{1}{2}$$

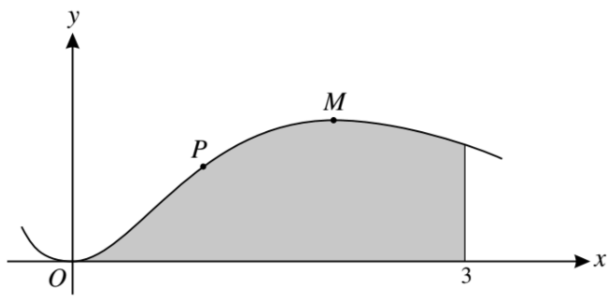
$$-\frac{1}{y} = \frac{\ln(\sin x) + \ln 2 - 1}{2}$$

$$y = \frac{-2}{\ln(2\sin x) - 1}$$

Integration

Area under curve

9709_s32_11 -q



The diagram shows the curve $y = x^2e^{-x}$.

(i) Show that the area of the shaded region bounded by the curve, the x-axis and the line $x = 3$ is equal to $2 - \frac{17}{e^3}$. [5]

$$\int_0^3 y\,dx = \int_0^3 \underbrace{x^2}_{u}\underbrace{e^{-x}}_{v}\,dx$$

ILATE

$u = x^2$, $v = e^{-x}$

$\frac{du}{dx} = 2x$, $\int v\,dx = -e^{-x}$

$$= u\int v\,dx - \int\left[\frac{du}{dx}\cdot\int v\,dx\right]dx$$

$$= x^2(-e^{-x}) - \int 2x(-e^{-x})\,dx$$

$$= -x^2e^{-x} + 2\int \underbrace{x}_{u_1}\,\underbrace{e^{-x}}_{v_1}\,dx$$

$$u_1 = x \quad , \quad v_1 = e^{-x}$$

$$\frac{du_1}{dx} = 1 \quad , \quad \int v_1\,dx = -e^{-x}$$

$$u_1\int v_1\,dx - \int\left[\frac{du_1}{dx}\cdot\int v_1\,dx\right]dx$$

$$= -x^2e^{-x} + 2\left[x(-e^{-x}) - \int 1\,(-e^{-x})\,dx\right]$$

$$= -x^2e^{-x} + 2\left[-xe^{-x} - e^{-x}\right]$$

$$= \left| -x^2e^{-x} - 2xe^{-x} - 2e^{-x} \right|_0^3$$

$$= \left|\left(-(3)^2e^{-3} - 2(3)e^{-3} - 2e^{-3}\right) - \left(-(0)^2e^0 - 2(0)e^0 - 2e^0\right)\right|$$

$$= \left|\left(-9e^{-3} - 6e^{-3} - 2e^{-3}\right) - (-2)\right|$$

$$= 2 - \frac{17}{e^3}$$

Integration with partial fractions

9709_s03_04 -q

(a) Show that $\int_3^4 \frac{3x}{(x+1)(x-2)}\,dx = \ln 5.$ [6]

$$\frac{3x}{(x+1)(x-2)} \equiv \frac{A}{x+1} + \frac{B}{x-2}$$

$$\frac{3x}{(x+1)(x-2)} \equiv \frac{A(x-2) + B(x+1)}{(x+1)(x-2)}$$

$$3x \equiv A(x-2) + B(x+1)$$

$$\begin{array}{l|l} x+1=0 & 3(-1) = A(-1-2) + B(-1+1) \\ x=-1 & -3 = -3A \\ & A = 1 \end{array}$$

$$\begin{array}{l|l} x-2=0 & 3(2) = A(2-2) + B(2+1) \\ x=2 & 6 = 3B \\ & B = 2 \end{array}$$

$$\int_3^4 \frac{1}{x+1} + \frac{2}{x-2}\, dx$$

$$\int_3^4 \frac{1}{x+1}\, dx + 2\int_3^4 \frac{1}{x-2}\, dx$$

$$\left|\ln(x+1)\right|_3^4 + 2\left|\ln(x-2)\right|_3^4$$

$$\left(\ln(4+1) - \ln(3+1)\right) + 2\left(\ln(4-2) - \ln(3-2)\right)$$

$$(\ln 5 - \ln 4) + 2(\ln 2 - \ln 1)$$

$$\ln\left(\frac{5}{4}\right) + 2\ln 2$$

$$\ln\left(\frac{5}{4}\right) + \ln 4$$

$$\ln 5$$

Integration with Rcos($\theta \pm \alpha$)

9709_s03_07 -q

(i) Express $\cos\theta + (\sqrt{3})\sin\theta$ in the form $R\cos(\theta - \alpha)$, where $R > 0$ and $0 < \alpha < \frac{1}{2}\pi$, giving the exact values of R and α. [3]

$\cos\theta + (\sqrt{3})\sin\theta \equiv R\cos(\theta - a)$

$\cos\theta + (\sqrt{3})\sin\theta \equiv R[\cos\theta\cos a + \sin\theta\sin a]$

$\cos\theta + (\sqrt{3})\sin\theta \equiv R\cos\theta\cos a + R\sin\theta\sin a$

$\cos\theta = R\cos\theta\cos a$, $(\sqrt{3})\sin\theta = R\sin\theta\sin a$

$1 = R\cos a$, $\sqrt{3} = R\sin a$

$\frac{R\sin a}{R\cos a} = \frac{\sqrt{3}}{1}$

$\tan a = \sqrt{3}$

$a = \tan^{-1}(\sqrt{3})$

$a = \frac{1}{3}\pi$

$R^2\cos^2 a = 1^2 = 1$

$R^2\sin^2 a = (\sqrt{3})^2 = 3$

$R^2\cos^2 a + R^2\sin^2 a = 1 + 3$

$R^2(\sin^2 a + \cos^2 a) = 4$

$R^2 = 4$

$R = 2$

$\cos\theta + (\sqrt{3})\sin\theta \equiv 2\cos(\theta - \frac{1}{3}\pi)$

(ii) Hence show that $\int_0^{\frac{1}{2}\pi} \frac{1}{(\cos\theta + (\sqrt{3})\sin\theta)^2}\, d\theta = \frac{1}{\sqrt{3}}$. [4]

$$\int_0^{\frac{1}{2}\pi} \frac{1}{(\cos\theta + \sqrt{3}\sin\theta)^2}\, d\theta$$

$$\int_0^{\frac{1}{2}\pi} \frac{1}{(2\cos(\theta - \frac{1}{3}\pi))^2}\, d\theta$$

$$\frac{1}{4}\int_0^{\frac{1}{2}\pi} \frac{1}{\cos^2(\theta - \frac{1}{3}\pi)}\, d\theta$$

$$\frac{1}{4}\int_0^{\frac{1}{2}\pi} \sec^2(\theta - \tfrac{1}{3}\pi)\, d\theta$$

$$\frac{1}{4}\left|\tan(\theta - \tfrac{1}{3}\pi)\right|_0^{\frac{1}{2}\pi}$$

$$\frac{1}{4}\left(\tan(\tfrac{1}{2}\pi - \tfrac{1}{3}\pi) - \tan(-\tfrac{1}{3}\pi)\right)$$

$$\frac{1}{4}\left(\tan\left(\frac{1}{6}\pi\right) - \tan\left(-\frac{1}{3}\pi\right)\right)$$

$$\frac{1}{4}\left(\frac{\sqrt{3}}{3} - (-\sqrt{3})\right)$$

$$\frac{1}{4}\left(\frac{4\sqrt{3}}{3}\right)$$

$$\frac{\sqrt{3}}{3} = \frac{1}{\sqrt{3}}$$

Integration by parts

9709_w03_07 -q

Use integration by parts to show that

$$\int_2^4 \ln x \, dx = 6\ln 2 - 2. \qquad [4]$$

$$\int_2^4 \ln x \, dx$$

$$\int_2^4 \underbrace{(\ln x)}_{u}\, \underbrace{1}_{v}\, dx$$

I
L
A
T
E

$u = \ln x$, $v = 1$

$\frac{du}{dx} = \frac{1}{x}$, $\int v\,dx = x$

$$u\int v\,dx - \int\left[\frac{du}{dx}\cdot\int v\,dx\right]dx$$

$$\ln x(x) - \int \frac{1}{x}(x)\,dx$$

$$x\ln x - \int 1\,dx$$

$$\left| x\ln x - x \right|_2^4$$

$$(4\ln 4 - 4) - (2\ln 2 - 2)$$

$$\ln 4^4 - 4 - \ln 2^2 + 2$$

$$\ln 256 - \ln 4 - 2$$

$$\ln\left(\frac{256}{4}\right) - 2$$

$$\ln 64 - 2$$

$$\ln 2^6 - 2$$

$$6\ln 2 - 2$$

Integration by substitution

9709_w31_10 -q

Let $I = \int_0^1 \frac{x^2}{\sqrt{(4 - x^2)}}\,dx$.

(i) Using the substitution $x = 2\sin\theta$, show that

$$I = \int_0^{\frac{1}{6}\pi} 4\sin^2\theta\,d\theta. \qquad [3]$$

$x = 2\sin\theta$	Lower Limit , Upper Limit
$\frac{dx}{d\theta} = 2\cos\theta$	$x = 0$, $x = 1$
	$0 = 2\sin\theta$, $1 = 2\sin\theta$
$dx = 2\cos\theta \, d\theta$	$0 = \sin\theta$, $\frac{1}{2} = \sin\theta$
	$\theta = 0$, $\theta = \frac{\pi}{6}$

$$\int \frac{x^2}{\sqrt{4-x^2}} \, dx$$

$$\int \frac{(2\sin\theta)^2}{\sqrt{4-(2\sin\theta)^2}} (2\cos\theta \, d\theta)$$

$$\int \frac{4\sin^2\theta}{\sqrt{4-4\sin^2\theta}} . 2\cos\theta \, d\theta$$

$$\int \frac{4\sin^2\theta}{\sqrt{4(1-\sin^2\theta)}} . 2\cos\theta \, d\theta$$

$$\int \frac{4\sin^2\theta}{\sqrt{4\cos^2\theta}} . 2\cos\theta \, d\theta$$

$$\int \frac{4\sin^2\theta}{2\cos\theta} . 2\cos\theta \, d\theta$$

$$\int_0^{\frac{\pi}{6}} 4\sin^2\theta \, d\theta$$

Complex numbers

Roots of equation

9709_w03_05 -q7

The equation $2x^3 + x^2 + 25 = 0$ has one real root and two complex roots.

(i) Verify that $1 + 2i$ is one of the complex roots. [3]

$$2x^3 + x^2 + 25 = 0$$

$$2(1+2i)^3 + (1+2i)^2 + 25 = 0$$

$$2(1^3 + {}^3C_1(1)^2(2i)^1 + {}^3C_2(1)(2i)^2 + {}^3C_3(1)^0(2i)^3) + (1 + 4i + 4i^2) + 25 = 0$$

$$2(1 + 6i + 12i^2 + 8i^3) + 4i^2 + 4i + 26 = 0$$

$$2 + 12i + 24i^2 + 16i^3 + 4i^2 + 4i + 26 = 0$$

$$2 + 12i + 24(-1) + 16i(-1) + 4(-1) + 4i + 26 = 0$$

$$0 = 0$$

(ii) Write down the other complex root of the equation. [1]

complex conjugate of $1 + 2i$

$1 - 2i$

Modulus and argument

9709_w03_08 -q10

The complex number w is given by $w = -\frac{1}{2} + \mathrm{i}\frac{\sqrt{3}}{2}$.

(i) Find the modulus and argument of w. [2]

$$w = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$$

$$R = \sqrt{\left(-\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$$

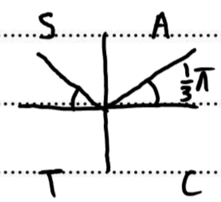
$$R = 1$$

arg w:

$$\alpha = \tan^{-1}\left(\frac{\frac{\sqrt{3}}{2}}{-\frac{1}{2}}\right)$$

$$\alpha = \tan^{-1}(-\sqrt{3})$$

$$\alpha = \frac{2}{3}\pi$$



Express u in the form x + iy

9709_w03_07 -q8

(a) The complex number z is given by $z = \frac{4-3i}{1-2i}$.

(i) Express z in the form $x + iy$, where x and y are real. [2]

$$\frac{4-3i}{1-2i} \quad \frac{1+2i}{1+2i}$$

$$\frac{(4-3i)(1+2i)}{(1-2i)(1+2i)}$$

$$\frac{4+8i-3i-6i^2}{(1)^2-(2i)^2}$$

$$\frac{4+5i-6(-1)}{1-4i^2}$$

$$\frac{10+5i}{1-4(-1)}$$

$$\frac{10+5i}{5}$$

$$2+i$$

Argand diagram

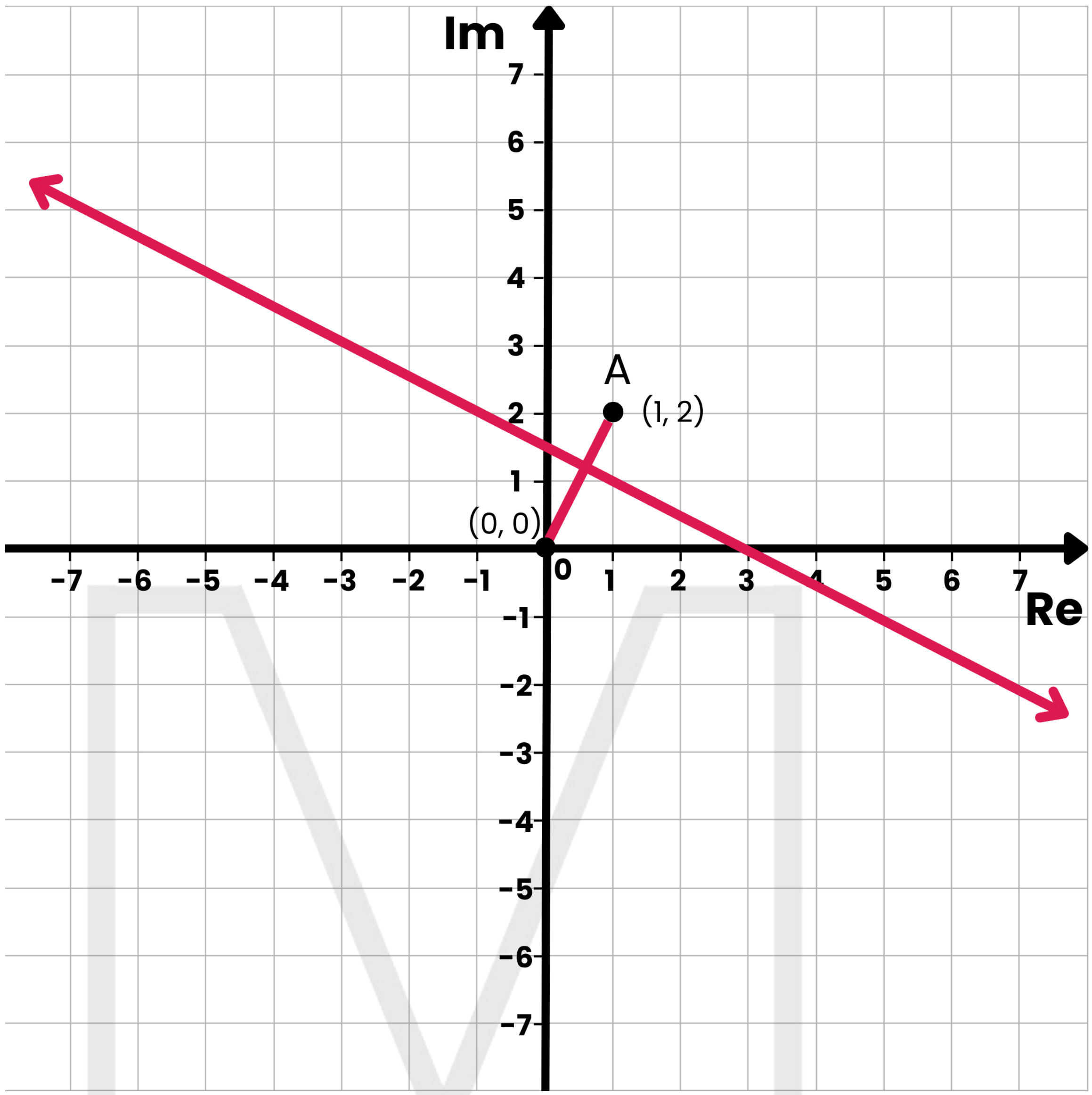
9709_w03_05 -q7

(iii) Sketch an Argand diagram showing the point representing the complex number $1 + 2i$. Show on the same diagram the set of points representing the complex numbers z which satisfy

$$|z| = |z - 1 - 2i|.$$ [4]

$$|z| = |z-1-2i|$$

$$|z-(0+0i)| = |z-(1+2i)|$$


Im
Re
A
(1, 2)
(0, 0)
0
-7 -6 -5 -4 -3 -2 -1 1 2 3 4 5 6 7
7 6 5 4 3 2 1 -1 -2 -3 -4 -5 -6 -7

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Solve the equation

9709_w03_09 -q7

(i) Solve the equation $z^2 + (2\sqrt{3})iz - 4 = 0$, giving your answers in the form $x + iy$, where x and y are real. [3]

$$z^2 + (2\sqrt{3})iz - 4 = 0$$

$$z = \frac{-(2\sqrt{3}i) \pm \sqrt{(2\sqrt{3}i)^2 - 4(1)(-4)}}{2(1)}$$

$$z = \frac{-2\sqrt{3}i \pm \sqrt{12i^2 + 16}}{2}$$

$$z = \frac{-2\sqrt{3}i \pm \sqrt{4}}{2}$$

$$z = \frac{-2\sqrt{3}i}{2} \pm \frac{2}{2}$$

$$z = 1 - \sqrt{3}i \quad , \quad z = -1 - \sqrt{3}i$$

MOJZA

A Note from Mojza

These notes for Mathematics (9709) have been prepared by Team Mojza, covering the content for A Level 2022-24 syllabus. The content of these notes has been prepared with utmost care. We apologise for any issues overlooked; factual, grammatical or otherwise. We hope that you benefit from these and find them useful towards achieving your goals for your Cambridge examinations.
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If you find any issues within these notes or have any feedback, please contact us at support@mojza.org.

Acknowledgements

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